1. The Exponential Function

The exponential function is defined by extending the Taylor series of $e^x$ from real values of $x$ to complex values:

$$e^z = 1 + z + z^2/2 + z^3/3! + \ldots$$  \hfill (1)

The partial sums are well defined since products and sums of complex numbers are well defined. With this definition, $e^z$ agrees with $e^x$ when $z$ is real. Furthermore, one can show that $e^z$ satisfies the properties of exponentials:

$$e^{z_1 + z_2} = e^{z_1}e^{z_2}, \quad e^{z_1 - z_2} = \frac{e^{z_1}}{e^{z_2}}, \quad (e^z)^n = e^{nz}, \quad \frac{d}{dz}e^z = e^z.$$

We showed in class that the definition (1) implies Euler’s formula: if $\theta$ is a real number, then

$$e^{i\theta} = \cos \theta + i \sin \theta$$  \hfill (2)

which is a point on the unit circle that subtends an angle $\theta$ with the real axis.

Every complex number $z$ in the complex plane is obtained by taking a point on the unit circle times its magnitude $r$. Thus, Euler’s formula enables us to write any number in the form

$$z = re^{i\theta} = r \cos(\theta) + ir \sin(\theta)$$  \hfill (3)

where $r$ is the modulus and $\theta$ is the argument. We’ll refer to this form as the polar representation of $z$ in terms of the coordinates $r, \theta$. In this form it is easy to take powers and roots of complex numbers.

**Example 1:** Plot the complex number $1 + i$ in the complex plane and find its polar representation.

**Solution:** By plotting the point we see that $\theta = \pi/4$ and $r = \sqrt{2}$. Thus $1 + i = \sqrt{2}e^{i\pi/4}$.

**Example 2:** Find the real and imaginary parts of $z = (1 + i)^{10}$

**Solution:** We use the polar representation $1 + i = \sqrt{2}e^{i\pi/4}$ and the properties of the exponential function to find that

$$z = (1 + i)^{10} = (\sqrt{2}e^{i\pi/4})^{10} = 2^5e^{i5\pi/2} = 2^5e^{i\pi} = 2^5i$$

so $Re(z) = 0$, $Im(x) = 2^5$.

Notice that taking a complex number $z = re^{i\theta}$ to a power

$$z^n = r^n e^{in\theta}$$

simply takes the power of the modulus and multiplies the argument by $n$. For example squaring a number squares the modulus and doubles the argument.

**Exercise:** Plot all integer powers of the complex number on the unit circle $(1 + i)/\sqrt{2}$.

Euler’s formula also enables us to identify the real and imaginary parts of $e^z$, where $z = x + iy$, as follows

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y) = e^x \cos y + ie^x \sin y.$$  \hfill (4)
so \( \text{Re}(z) = e^x \cos y \) and \( \text{Im}(z) = e^x \sin y \).

**Example 3:** Find the real and imaginary parts of \( z = e^{1+ i} \).

**Solution:** \( z = e^{1+ i} = e^1 e^i = e (\cos 1 + i \sin 1) \), so \( \text{Re}(z) = e \cos 1 \) and \( \text{Im}(z) = e \sin 1 \).

Note that the argument of a complex number is not uniquely defined since you can add any multiple \( 2n\pi \) of \( 2\pi \), where \( n \) is any positive or negative integer, to \( \theta \) and get the same points. That is, if \( z = re^{i\theta} \) then also

\[
z = re^{i(\theta + 2n\pi)}
\]

For example, the number \( 1 = e^{0i} = e^{(0+2n\pi)i} = e^{2n\pi i} \). We’ll use this to find all complex roots of equations. In particular, all **roots of unity**.

**Example 4:** Find all complex roots of \( z^3 = 1 \).

**Solution:** By taking the modulus on both sides we find that \( |z|^3 = |z|^4 = 1 \) so \( |z| = 1 \) so \( z \) must be a point on the unit circle,

\[
z = e^{i\theta}
\]

By plugging this into the equation and writing \( 1 = e^{2n\pi i} \) we get that

\[
e^{i3\theta} = e^{2n\pi i}
\]

Now we can equate powers to get

for \( n = 0 \) : \( \theta = 0 \)

for \( n = 1 \) : \( \theta = 2\pi/3 \)

for \( n = 2 \) : \( \theta = 4\pi/3 \)

Any other values of \( n \) only give values of \( \theta \) that agree with one of the above to within a multiple of \( 2\pi \). So the 3 distinct cubic roots of unity are

\[
z = 1, e^{2\pi i/3}, e^{4\pi i/3}.
\]

2. The complex logarithm

Once the exponential function is defined for complex numbers we can define the logarithm of complex numbers, as follows. Let \( \text{Log} r \) denote the natural logarithm of a positive real number \( r \), as used in calculus. Write \( z = re^{i\theta} \). For any \( z \neq 0 \), the logarithm of \( z \) is defined by

\[
\log z = \log(re^{i\theta}) = \text{Log} r + i\theta \tag{5}
\]

With this definition, the logarithm satisfies the familiar properties of the natural logarithm of a positive real number: \( \log(z_1 z_2) = \log z_1 + \log z_2 \), \( \log(z_1/z_2) = \log z_1 - \log z_2 \).

**Nonuniqueness:** Note that the logarithm of a complex number is not unique, since the argument \( \theta \) is defined only up to multiples of \( 2\pi \), since \( z = re^{i\theta} = re^{i(\theta + 2n\pi)} \), so

\[
\log z = \text{Log} r + i(\theta + 2n\pi) \tag{6}
\]

**Example 5:** \( \log 1 = \log(e^{i(0+2\pi)}) = i2n\pi \) for all integers \( n \).
\[ e^{\log z} = z \] (7)

regardless of what multiple of 2\(\pi\) is added to the argument \(\theta\). To see this, write \(z = re^{i\theta} = re^{i(\theta + 2n\pi)}\) and \(\log z = \log r + i(\theta + 2n\pi)\). Then \(e^{\log z} = e^{\log r + i(\theta + 2n\pi)} = e^{\log r}e^{i2n\pi} = re^{i\theta} = z\) since \(e^{i2n\pi} = 1\). However, it is not true that \(\log(e^z) = z\), since the logarithm has an infinite number of values for any given value of \(z\).

Once the logarithm is defined, we can compute the value of the exponent \(z^c\) for any complex exponent \(c\), using the following equation (6), which holds in view of equation (5),

\[ z^c = e^{c\log z} \] (8)

**Example 6**: \(i^2 = e^{2i\log i} = e^{2i(\pi/2 + 2n\pi)} = e^{-(\pi+4n\pi)} = e^{-(4n+1)\pi}\) for all integers \(n\). As can be seen from this example, powers of the form \(z^c\) are multivalued. This follows from (6) and the fact that the logarithm of a complex number is multivalued.

**Example 7**: Remember the special case also treated in the book where \(c\) is a rational number. In particular, if \(c = 1/n\), then

\[ z^{1/n} = (re^{i\theta + 2mi\pi})^{1/n} = r^{1/n}e^{i\theta/n + 2mi\pi/n} \]

which gives you \(n\) different values corresponding to \(m = 0 \ldots n - 1\).

One way to avoid the nonuniqueness of the logarithm is to restrict the value of the argument \(\theta\) to a principal branch of the logarithm, for which \(\theta \in (-\pi, \pi]\). This principal branch is called \(\text{Log } z\), and agrees with the natural logarithm for real numbers that we are familiar with. Hence

\[ \text{Log } z = \text{Log}(re^{i\theta}) = \text{Log } r + i\theta \quad \text{where } \theta \in (-\pi, \pi] \] (9)

**Example 8**: \(\text{Log1} = \text{Log}(e^{i(0 + 2n\pi)}) = 0\) (this is the only argument that lies in \((-\pi, \pi)\))

**Example 9**: \(\text{Log}(2i) = \text{Log}(2e^{i(\pi/2 + 2n\pi)}) = \text{Log}2 + i\pi 3/2\)

3. Deriving Trig identities using Euler’s Formula

Euler’s formula and the properties of exponentials make it easy to derive several of the trigonometric identities we have been using.

**Example 10**: Use exponentials to derive formulas for \(\sin 2x\) and \(\cos 2x\).

**Solution**: Start with the identity \(e^{2ix} = (e^{ix})^2\). Using Euler’s formula we rewrite this identity as

\[ \cos 2x + i\sin 2x = (\cos x + i\sin x)^2 = \cos^2 x - \sin^2 x + 2i\sin x \cos x \]

By equating real and imaginary parts on the left and right hand side of this equation we get

\[ \cos 2x = \cos^2 x - \sin^2 x, \quad \sin 2x = 2\sin x \cos x \]

4. Integration using Complex Variables

Complex variables can also be used to determine integrals that are tedious to evaluate using conventional methods. For example, consider integrals of the form \(\int e^{at} \cos bt \, dt\). Recall how complicated
this is to integrate using integration by parts. Here it is shown that the use of complex variables can sometimes reduce the complexity of the integration of such functions. You will need to use Euler’s equation and the definition of equality for complex numbers – namely that if \( z_1 = z_2 \) then \( \text{Re}(z_1) = \text{Re}(z_2) \) and \( \text{Im}(z_1) = \text{Im}(z_2) \).

We first define what the integral of a complex function means. Consider a complex valued function \( w(t) = u(t) + iv(t) \), where \( u \) and \( v \) are real-valued function and \( t \) is a real variable. (That is, \( u \) and \( v \) are the real and imaginary parts of \( w \).) Then the integral of \( w \) is defined by

\[
\int_a^b w(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt \tag{10}
\]

**Example 11:** Evaluate the integral \( \int_0^{\pi/3} e^{2it} \) dt.

**Solution:**
\[
\int_0^{\pi/3} e^{2it} dt = \int_0^{\pi/3} \cos 2t + i \sin 2t dt = \int_0^{\pi/3} \cos 2t dt + i \int_0^{\pi/3} \sin 2t dt
= \left[ \frac{\sin 2t}{2} \right]_0^{\pi/3} + i \left[ -\frac{\cos 2t}{2} \right]_0^{\pi/3} = \frac{\sin(2\pi/3)}{2} - i \left( \frac{\cos(2\pi/3) - 1}{2} \right) = \frac{\sqrt{3}}{4} + \frac{3i}{4}
\]

We now illustrate that some of the integrals we computed earlier this semester are evaluated quite easily if we use complex variables instead. The following examples use the fact that

\[
\int e^{zt} dt = \frac{1}{z} e^{zt} + C \quad \text{and} \quad \int \frac{1}{z} dt = \frac{1}{z} \log(z0t) + C . \tag{11}
\]

These identities follow from the fact that \( \frac{d}{dz} e^z = e^z \) and \( \frac{d}{dz} \log z = \frac{1}{z} \) together with the Chain Rule.

**Example 12:** Evaluate the integral \( \int e^{(a+ib)t} \) dt. What do you learn by equating real and imaginary parts on the left and right hand sides of the resulting equation?

**Solution:**
\[
\int e^{(a+ib)t} dt = \frac{1}{a+ib} e^{(a+ib)t} + C .\]
Equating real and imaginary parts gives

\[
\int e^{at} \cos bt dt = \frac{e^{at}}{a^2 + b^2} (a \cos bt + b \sin bt) + C_1 ,
\]

\[
\int e^{at} \sin bt dt = \frac{e^{at}}{a^2 + b^2} (a \sin bt - b \cos bt) + C_2 ,
\]

Notice that to compute these two integrals without complex variables, we would have to use integration by parts twice. These integrals occur frequently in mechanical systems. The functions \( e^{at} \cos bt, e^{at} \sin bt \) represent oscillating functions whose magnitude grows or decays exponentially.

**Example 13:** Evaluate the integral \( \int_0^{2\pi} \sin kx \sin mx dx \) using complex variables, where \( k, m \) are positive integers.

**Solution:** As we saw in class, a consequence of Euler’s formula is that we can write \( \sin x \) and \( \cos x \) in terms of complex exponentials:

\[
\sin x = \frac{e^{ix} - e^{-ix}}{2i} , \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}
\]

Substituting these identities into the integral we obtain
\[
\int_0^{2\pi} \frac{e^{ikx} - e^{-ikx}}{2i} \left( \frac{e^{imx} - e^{-imx}}{2i} \right) \, dx = \frac{1}{4} \int_0^{2\pi} e^{i(k+m)x} - e^{i(k-m)x} - e^{i(m-k)x} + e^{i(-m-k)x} \, dx
\]

\[
= \begin{cases} 
- \frac{1}{4}(0 - 2\pi - 2\pi + 0) & \text{if } k = m \\
0 & \text{if } k \neq m \\
\pi & \text{if } k = m \\
0 & \text{if } k \neq m 
\end{cases}
\]

These results follow in view of Homework Pb 9 below. (Note that \(k+m \neq 0\) since \(k, m > 0\) and \(k - m = 0 \iff k = m\).) Notice that we computed these integrals in the past using Table 2, p.515, in the book. The integration using complex variables is more natural.

Example 14: Evaluate the integral \(\int \frac{1}{1+x^2} \, dx\) using partial fractions with complex variables.

Solution:
\[
\int \frac{1}{1+x^2} \, dx = \frac{1}{2} \int \frac{1}{1+ix} + \frac{1}{1-ix} \, dx
\]
\[
= \frac{1}{2i} \left( \frac{1}{i} \log(1+ix) - \frac{1}{i} \log(1-ix) \right) + C
\]
\[
= \frac{1}{2i} \log \frac{1+ix}{1-ix} + C = \frac{1}{2i} \log \frac{re^{i\theta}}{re^{-i\theta}} + C
\]
\[
= \frac{1}{2i} \log(e^{2i\theta}) + C = \frac{1}{2i} 2i\theta + C = \theta + C
\]

This follows if one writes \(1 + ix = re^{i\theta}\) and \(1 - ix = re^{-i\theta}\), where \(r = \sqrt{1+x^2}\) and \(\theta = \tan^{-1}x\), as shown in the Figure. This example illustrates how the arctan comes up naturally as the antiderivative of \(\frac{1}{1+x^2}\) through the logarithm.

Problems: Part I

Powers of complex numbers:
1. Write the following numbers in the form \(re^{i\theta}\) and in the form \(a+ib\): (a) \((1+i)^{20}\), (b) \((1-\sqrt{3}i)^5\), (c) \((2\sqrt{3}+2i)^5\), (d) \((1-i)^8\).
2. Find all solutions to the given equation. Sketch the roots in the complex plane.
   (a) \(z^8 = 1\) (the eighth roots of 1)
   (b) \(z^3 = i\) (the cube roots of \(i\))
   (c) \(z^5 = 32\) (the fifth roots of 32)
   (d) \(z^3 = 1+i\) (the cube roots of \(1+i\))

Problems: Part II

Exponentials and Logarithms:
1. Show that: (a) \(e^{2+3i} = -e^2\) (b) \(e^{2+i \pi} = \sqrt{2}(1+i)\) (c) \(e^{z+i\pi} = -e^z\)
2. Find the real and imaginary parts of \(a) \log(2+i) \quad b) \log(1+i) \quad (t\ is\ real) \quad e) \log(1-i)\)
3. Write the following complex numbers in the form \(a+ib\) where \(a,b\ are\ real\).
   (a) \(\log e\) (b) \(\log i\) (c) \(\log(-1+\sqrt{3}i)\) (d) \(\log(-e)\) (e) \(\log(1-i)\)
4. Find all the roots to the equation \( \log z = (\pi/2)i \).

5. Write the following complex numbers in the form \( re^{i\theta} \) where \( r, \theta \) are real.
   (a) \((1 + i)^t \)
   (b) \((-1)^\pi \)

6. Find all roots to the equation \( z^5 = 1 \), and plot the solutions in the complex plane.

**Trigonometric Identities:**

7. Derive the formulas for \( \sin(a + b) \) and \( \cos(a + b) \) using complex variables. (Hint: look at the real and imaginary parts of the equation \( e^{i(a+b)} = e^{ia}e^{ib} \)).

**Integration:**

8. Evaluate the following integrals either with the definition (8) on p.2 or using Eq (9) on p.3.
   (a) \( \int_0^1 (1 + it^2) \, dt \)
   (b) \( \int_0^{\pi/4} e^{it} \, dt \)

9. (a) Show that if \( j \) is an integer \( \int_0^{2\pi} e^{ijx} \, dx = \begin{cases} 0 & \text{if } j \neq 0 \\ 2\pi & \text{if } j = 0 \end{cases} \)
   (b) Show that if \( k \) and \( n \) are integers, \( \int_0^{2\pi} e^{ikx}e^{inx} \, dx = \begin{cases} 0 & \text{if } k+n \neq 0 \\ 2\pi & \text{if } k+n = 0 \end{cases} \)
   (Hint: Use the properties of exponentials \( e^x e^y = e^{x+y} \))

10. (a) Show that \( \int_0^{2\pi} \sin kx \cos mx \, dx = 0 \), where \( k, m \) are positive integers.
    (b) Show that \( \int_0^{2\pi} \cos kx \cos mx \, dx = \begin{cases} \pi & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases} \), where \( k, m \) are positive integers.