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On spectra of linearized operators for Keller–Segel models of chemotaxis[☆]

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ABSTRACT

We consider the phenomenon of collapse in the critical Keller–Segel equation (KS) which models chemotactic aggregation of micro-organisms underlying many social activities, e.g. fruiting body development and biofilm formation. Also KS describes the collapse of a gas of self-gravitating Brownian particles. We find the fluctuation spectrum around the collapsing family of steady states for these equations, which is instrumental in the derivation of the critical collapse law. To this end we develop a rigorous version of the method of matched asymptotics for the spectral analysis of a class of second order differential operators containing the linearized Keller–Segel operators (and as we argue linearized operators appearing in nonlinear evolution problems). We explain how the results we obtain are used to derive the critical collapse law, as well as for proving its stability.

1. Introduction

Phenomena of blowup and collapse in nonlinear evolution equations are hard to simulate numerically and the rigorous theory, or at least a careful analysis, is pertinent here. The recent years witnessed a tremendous progress in developing of such theories. We can now describe the shape of blowup profile and contraction law in Yang–Mills, σ-model, nonlinear Schrödinger and heat equations [1–9].[1] Yet, after 40 years of intensive research and important progress, we still cannot give a rigorous description of collapse in the Keller–Segel equations modeling chemotaxis. (See [11–19] for some recent works, [20], for a nice discussion of the subject, and [21–25] for reviews.)

This is not say that the Keller–Segel equations are harder than Yang–Mills, σ-model and nonlinear Schrödinger equations, they are not, but neither are they less important. They model chemotaxis, which is a directed movement of organisms in response to the concentration gradient of an external chemical signal ([26], see also [27]). The chemical signals can come from external sources or they can be produced by the organisms themselves. The latter situation leads to aggregation of organisms and to the formation of patterns and is the case modeled by the Keller–Segel equations. Chemotaxis is believed to underly many social activities of micro-organisms, e.g. social motility, fruiting body development, quorum sensing and biofilm formation. A classical example is the dynamics and the aggregation of an *Escherichia coli* colony under the starvation condition [11]. Another example is the *Dictyostelium* amoeba, where single cell bacterivores, when challenged by adverse conditions, form multicellular structures of $\sim 10^5$ cells [28,29]. Also endothelial cells of humans react to vascular endothelial growth factor to form blood vessels through aggregation [30].

Assuming that the organism population is large and the individuals are small relative to both the domain, $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) and the typical distance between the organisms, one can derive in the mean-field approximation the Keller–Segel system governing the organism density $\rho : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ and chemical concentration $c : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ [26,31]. As the chemical diffuses much faster than organisms, one makes a simplifying assumption of instantaneous interaction (adiabatic assumption) which, after rescaling and a minor simplification, leads the Keller–Segel equations to the form

$$\begin{cases} \partial_t \rho = \Delta\rho - \nabla\cdot(\rho\nabla c), \\ 0 = \Delta c + \rho, \end{cases} \tag{1}$$

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1 Numerical simulations for these equations failed until the compression rate was derived analytically, see [6,5,10].

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with ρ and c satisfying the no-flux Neumann boundary conditions. The Eqs. (1) appear also in the context of stellar collapse (see [32–35]); similar equations—the Smoluchowski or nonlinear Fokker—Planck equations—models non-Newtonian complex fluids (see [36–39]).

Arguably, the most interesting feature of the Keller–Segel equations is that they can develop, in finite time, infinite mass at a point in space. As argued below, the 'collapsing' profile and contraction law have a universal (close to self-similar) form, independent of particulars of initial configurations and, to a certain degree, of the equations themselves, and can be associated with chemotactic aggregation. Though the equations are rather crude and unlikely to produce patterns one observes in nature or experiments, the collapse phenomenon could be useful in verifying assumptions about biological mechanisms.[2]

We now concentrate on the (energy) critical case of $d = 2$ and $\Omega = \mathbb{R}^2$. It was shown in [43,44] that solutions of (1) with the mass

$$M := \int_{\mathbb{R}^2} \rho_0 \, dx > 8\pi$$

blow up in finite time. Ref. [13] exhibited blowup solutions with explicit blowup rate and explicit asymptotics, which was confirmed in [45,46] by a different technique relying on results of the present paper. However, the problem of describing the dynamics of blowup, i.e. blowup rate and profile for an open set of initial conditions is still open. As is shown below, this paper makes a considerable progress toward its solution.

Of critical importance here are the following key properties of the Eq. (1):

- It is invariant under the scaling transformations $\rho(x,t) \to \frac{1}{\lambda^2}\rho\left(\frac{1}{\lambda}x, \frac{1}{\lambda^2}t\right)$ and $c(x,t) \to c\left(\frac{1}{\lambda}x, \frac{1}{\lambda^2}t\right)$.
- It has the static solution, $R(x) := \frac{8}{(1+|x|^2)^2}$, $C(x) := -2\ln(1 + |x|^2)$.
- The total 'mass' is conserved: $\int \rho(x,t)dx = \text{const.}$[3]

The stationary solution $R(x)$ has the total mass $\int R(x)dx = 8\pi$, which is exactly the sharp threshold between global existence and singularity development in solutions to (1), mentioned above.

The properties above yield that (1) has in fact the family of static solutions $\lambda^{-2}R(x/\lambda)$, $C(x/\lambda)$, $\lambda > 0$,[4] and suggest a likely scenario of collapse: sliding along this family in the direction of $\lambda \to 0$. Indeed, we *conjecture* that, like in Struwe's result [51] for equivariant wave maps from the Minkovskii space–time, $\mathbb{M}^{2+1}$, to the 2-sphere, S^2, for any solution, $\rho(x,t)$, of (1), collapsing up at time T, there are sequences $\lambda_i \to 0$ and $t_i \to T$, s.t. $\rho(\lambda_i y, t_i)$ converges to the stationary solution $R(y)$, as $i \to \infty$. Thus the most interesting and natural initial conditions for (1) are those close to the manifold $\{R_\lambda(x)|\lambda > 0\}$.

This discussion brings us to the first step of the theory of collapse in the Keller–Segel system—determining the low-lying spectrum of fluctuations around the family $R_\lambda(x)$. This would determine whether this family is stable. In this paper we find this spectrum and to do this we develop a rigorous version of the method of matched asymptotics.

Now we discuss, following [48], a natural approach to this problem. Since the blowup profile is expected to be radially symmetric, it is natural to start with radially symmetric solutions. In this case, the system (1), which consists of coupled parabolic and elliptic PDEs, is equivalent to a single PDE. Indeed, the change of the unknown, by passing from the density, $\rho(x,t)$, to the normalized mass,

$$m(r,t) := \frac{1}{2\pi}\int_{|x|\le r} \rho(x,t)\, dx,$$

of organisms contained in a ball of radius r, discovered by [52,20], maps two Eqs. (1) into a single equation

$$\partial_t m = \Delta_r^{(0)} m + r^{-1} m \partial_r m, \tag{2}$$

on $(0,\infty)$ (with initial condition $m_0(r) := \frac{1}{2\pi}\int_{|x|\le r}\rho_0(x)\,dx$). Here $\Delta_r^{(n)}$ is the n-dimensional radial Laplacian, $\Delta_r^{(n)} := r^{-(n-1)}\partial_r r^{n-1}\partial_r = \partial_r^2 + \frac{n-1}{r}\partial_r$. Thus, in the radially symmetric case, (1) is equivalent to (2).

The Eq. (2) has the following key properties, inherited from the corresponding properties of (1):

- It is invariant under the scaling transformations $m(r,t) \to m\left(\frac{1}{\lambda}r, \frac{1}{\lambda^2}t\right)$.
- It has the static solution $\chi(r) := \frac{4r^2}{1+r^2}$.
- The total 'mass' is conserved: $2\pi \lim_{r\to\infty} m(r,t) = \int \rho(x,t)dx = \text{const.}$

As in the case of (1), the properties above yield the manifold of static solutions $\mathcal{M}_0 := \{\chi(r/\lambda) \mid \lambda > 0\}$ and suggest a likely scenario of collapse: sliding along $\mathcal{M}_0$ in the direction of $\lambda \to 0$. To analyze the collapse, we pass to the reference frame collapsing with the solution, by introducing the adaptive blowup variables,

$$m(r,t) = u(y,\tau), \quad \text{where } y = \frac{r}{\lambda} \text{ and } \tau = \int_0^t \frac{1}{\lambda^2(s)}\, ds,$$

where $\lambda : [0,T) \to [0,\infty)$, $T > 0$, is a positive differentiable function (*compression* or *dilatation* parameter), s.t. $\lambda(t) \to 0$ as $t \uparrow T$. The advantage of moving to blowup variables is that the function u is expected to have bounded derivatives and the blowup time is eliminated from consideration (it is mapped to ∞). Writing (2) in blowup variables, we find the equation for the rescaled mass function

$$\partial_\tau u = \Delta_y^{(0)} u + y^{-1} u \partial_y u - a y \partial_y u, \tag{3}$$

where $a := -\dot{\lambda}\lambda$.

To investigate stability properties of the rescaled stationary solution $\chi(y)$, we decompose solutions $u(y,\tau)$ of Eq. (3) into the main term, $\chi(y)$, and the fluctuation $\phi(y,\tau)$, $u(y,\tau) = \chi(y) + \phi(y,\tau)$. Substituting this decomposition into (3) gives the equation for the fluctuation ϕ,

$$\partial_\tau \phi = -L_a \phi + F_a + N(\phi), \tag{4}$$

where the forcing and nonlinear terms are $F_a := -\frac{8ay^2}{(1+y^2)^2}$ and $N(\phi) := \frac{1}{y}\phi\partial_y\phi$, and the linear operator, L_a is given by

$$L_a := -\Delta^{(4)} - \frac{8}{(1+y^2)^2} + \frac{4}{y(1+y^2)}\partial_y + ay\partial_y. \tag{5}$$

[2] There are numerous refinements of the Keller–Segel equations, e.g. taking into account finite size of organisms [40–42] preventing the complete collapse, which model the chemotaxis more precisely. We believe techniques we outline and develop here can be applied to these models as well.

[3] Another important property of (1) that it is a gradient system with the (free) energy $F(\rho) = \int_{\mathbb{R}^2}(\frac{1}{2}\rho\,\Delta^{-1}\rho + \rho\ln\rho)\,dx$, which plays the key role in other papers, is not used in our approach.

[4] It seems this family was discovered in [47]. It is shown in [48] that belongs to the two parameter family $R_\lambda^{(\mu)}(x) := R^{(\mu)}(r/\lambda)$, where $R^{(\mu)}(x) := 2(\mu-2)^2\frac{|x|^{\mu-4}}{(1+|x|^{\mu-2})^2}$, $\mu > 2$. Our case is $\mu = 4$. If $2 < \mu < 4$, then the mass at the origin is non-zero, and if $\mu > 4$, then the mass at the origin is negative and hence the static solution is not physical. For $\mu = 4$, due to the sharp logarithmic Hardy–Littlewood–Sobolev inequality, these static solutions unique and minimize the free energy, $F(\rho) = \int_{\mathbb{R}^2}(\frac{1}{2}\rho\,\Delta^{-1}\rho + \rho\ln\rho)\,dx$, for the fixed mass $\int\rho = \mu$ [49,50]. We conjecture that the same is true for $2 < \mu < 4$.

An important fact here is that the operator L_a is self-adjoint on the space $L^2(\mathbb{R}_+, \gamma_a(y)y^3dy)$, where $\gamma_a^{-1/2}(y) = \chi(y)e^{\frac{a}{4}y^2}$, with the inner product $\langle f, g\rangle := \int_0^\infty f(y)g(y)\,\gamma_a(y)y^3dy$. One can check the self-adjointness of L_a directly or use the unitary map

$$\phi(y) \to \gamma_a^{1/2}(y)\phi(y), \tag{6}$$

from $L^2([0,\infty), \gamma_a(y)y^3dy)$ to $L^2([0,\infty), y^3dy)$, which maps this operator L_a into the operator $\mathcal{L}_a := \gamma_a^{1/2}L_a\gamma_a^{-1/2}$, acting on the space $L^2([0,\infty), y^3dy)$. The latter operator can be explicitly computed to be

$$\mathcal{L}_a := -\Delta^{(4)} - \frac{8}{(1+y^2)^2} + \frac{1}{4}a^2y^2 + \frac{2a}{1+y^2} - 2a. \tag{7}$$

This operator is of the Schrödinger type with the real continuous potential tending to ∞ as $y \to \infty$. Therefore, by standard arguments (see e.g. [53]), it is self-adjoint and its spectrum is purely discrete. Hence L_a is self-adjoint on the space $L^2([0,\infty), \gamma_a(y)y^3dy)$ and has purely discrete spectrum as well. Going through with our analysis shows that $a(\tau) \to 0$ as $\tau \to \infty$, which actually complicates the problem and which tells us that the collapse is slower than parabolic one, $\lambda(t) = \sqrt{a_0(T-t)}$, for which $a(\tau) = -\lambda(t)\dot\lambda(t)$ is a constant (say, a_0).

Now, it is clear that the stability of the profile $\chi(y)$ is determined largely by the spectrum of the operator $\mathcal{L}_a$. If the operator $\mathcal{L}_a$ has strictly positive spectrum, one expect the fluctuations ϕ will die out as $\tau \to \infty$ and consequently the solution of (3) will tend to $\chi(y)$, while the solution of (2) will approach $\chi(r/\lambda(t))$. On the other hand, if the operator $\mathcal{L}_a$ has negative eigenvalues then one expects instability. The latter though is always the case, since the equations have a negative scaling mode (for a fixed parabolic scaling it is connected to possible variation of the blowup time).[5]

If the number of negative eigenvalues is finite, say k, then one either goes to an invariant manifold theory and constructs the central-unstable manifold or, uses the (related) modulation theory and embeds $\chi(r/\lambda)$ into a k-parameter family of almost solutions, say $\chi_p(r/\lambda)$, where p stands for the $k-1$ parameters (with λ, or a, counted as the first parameter), chosen so that the tangent space of the deformation (or almost center-unstable) manifold $\mathcal{M} := \{\chi_p(r/\lambda) \mid \lambda > 0,\ p\}$ at $\chi_p(y)$ is equal approximately to the eigenspace of negative and (almost) zero spectrum of $\mathcal{L}_a$. Then we can choose the parameters $p = p(\tau)$ and $a = a(\tau)$ (or $\lambda = \lambda(\tau)$), so that the solution $u(y,\tau)$ can be decomposed as

$$u(y,\tau) = \chi_{p(\tau)}(y) + \phi(y,\tau), \tag{8}$$

with the fluctuation $\phi(y,\tau)$ orthogonal to the tangent space of $\mathcal{M}$ at $\chi_p(y)$, $\langle \partial_p\chi_{p(\tau)}(\cdot), \phi(\cdot,\tau)\rangle = 0$, and therefore (approximately) orthogonal to the negative and almost zero spectrum eigenfunctions of $\mathcal{L}_a$. If we find such a deformation, $\chi_{p(\tau)}(y)$, then the stability is restored and the solution to (3) approaches this family as $\tau \to \infty$. The latter is a big if and this is where the understanding the negative and almost zero spectrum eigenfunctions of $\mathcal{L}_a$ helps. (One should keep in in mind that the deformation of $\chi(y)$ will change the linearized operator $\mathcal{L}_a$ and the gauge transformation (6), but both can be easily handled.)

Theorem 1 implies that the operator $\mathcal{L}_a$ of the Keller–Segel system has one negative (corresponding to the scaling mode mentioned above) and one near zero eigenvalue, while the third eigenvalue, $2a + \frac{2a}{\ln\frac{1}{a}} + O\left(a\ln^{-2}\frac{1}{a}\right)$, is positive, but vanishing as $a \to 0$. (It also isolates the correct perturbation (adiabatic) parameter-$\frac{1}{\ln\frac{1}{a}}$.) Hence we have to construct a one-parameter deformation of $\chi(y)$ (remember that λ, or a, is counted as a parameter). For technical reasons it is convenient to use a two-parameter family, $\chi_{bc}(y)$, with an extra relation between the parameters a, b and c. In [48] we *choose the family*

$$\chi_{bc}(y) := \frac{4by^2}{c+y^2}, \tag{9}$$

with $b > 1$ and both parameters b and c are close to 1. Note that this family evolves on a different spatial scale than $\phi(y,\tau)$ in (8), as it can rewritten as $\chi_{bc}(y) = \chi_{\frac{b}{c},1}(\frac{y}{\sqrt{c}})$. The contraction law is obtained by using the orthogonality condition, $\langle \partial_{bc}\chi_{bc}, \phi\rangle = 0$. The latter is equivalent to two conditions,

$$\partial_\tau\langle \partial_{bc}\chi_{b(\tau)c(\tau)}(\cdot), \phi(\cdot,\tau)\rangle = 0 \tag{10}$$

and $\langle \partial_{bc}\chi_{b(\tau)c(\tau)}(\cdot), \phi(\cdot,\tau)\rangle|_{t=0} = 0$. We evaluate (10) by using the evolution equation, $\partial_\tau\phi = -L_{abc}\phi + F_{abc} + N(\phi)$, for ϕ, similar to (4), which follows by plugging the decomposition (8) into (3), and the explicit expressions,

$$\begin{aligned} \zeta_{bc1}(y) &:= \frac{1}{4}\partial_b\chi_{bc}(y) = \frac{y^2}{c+y^2},\\ \zeta_{bc2}(y) &:= \frac{1}{4b}\partial_c\chi_{bc}(y) = \frac{y^2}{(c+y^2)^2}, \end{aligned} \tag{11}$$

for the tangent vectors to the manifold $\mathcal{M}$. (The scaling mode $\zeta_{bc0}(y) := \frac{1}{8bc}y\partial_y\chi_{bc}(y) = \frac{y^2}{(c+y^2)^2}$ a multiple of $\zeta_{bc2}(y)$ which confirms that one of the parameters is superfluous.) This gives ordinary differential equations for a, b and c with higher order terms depending on ϕ:

$$\begin{cases} c_\tau + S(\phi,a,b,c)a_\tau = 2a - \dfrac{4d}{\ln\left(\frac{1}{a}\right)} + R(\phi,a,b,c),\\ \dfrac{d_\tau}{a} - S(\phi,a,b,c)a_\tau = -\dfrac{2d}{\ln\left(\frac{1}{a}\right)} + R(\phi,a,b,c), \end{cases} \tag{12}$$

where $d := b-1$, $|S(\phi,a,b,c)| \lesssim \frac{\|\phi\|_{L^2}}{a^{d+1}\ln(\frac{1}{a})}$ and $|R(\phi,a,b,c)| \lesssim \frac{a}{\ln^2(\frac{1}{a})}\frac{1}{\ln(\frac{1}{a})}[d\|\phi\|_{L^2} + \|(1+y^2)^{-1}\phi\|_{L^2}^2]$. These higher order terms are controlled by using a differential inequality for the Lyapunov functional $\phi \mapsto \|\phi\|_{L^2}^2$ and the inequality $\langle\phi, \mathcal{L}_{abc}\phi\rangle \geq 2a\|\phi\|^2$, which follows from our result below.[6] (One can also use higher order Lyapunov functionals like $\langle\phi, \mathcal{L}_{abc}^k\phi\rangle$, $k \geq 1$. The fact that the positive eigenvalues of L_{abc} vanish makes estimating ϕ a delicate matter.) Finally, we choose a relation between a, b and c so as to eliminate large terms in the corresponding vector fields, namely, $d = \frac{1}{2}a\ln(\frac{1}{a})$, This leads, in the leading order, to the differential equation

$$a_\tau = -\frac{2a^2}{\ln\left(\frac{1}{a}\right)}, \tag{13}$$

whose solutions, in the leading order, are $\frac{1}{a}\left(\ln\frac{1}{a} + O(1)\right) = 2\tau$ which results in $\ln\frac{1}{a(\tau)} = \ln 2\tau - \ln\ln 2\tau + \frac{\ln\ln 2\tau}{\ln 2\tau} + O\left(\frac{1}{\ln 2\tau}\right)$. Recalling that $\lambda(t)\dot\lambda(t) = -a(\tau)$ and using that $\lambda(t)\dot\lambda(t) = \lambda(\tau(t))^{-1}\partial_\tau\lambda(\tau(t))$, we obtain that $-\ln\lambda(\tau) = \int^\tau a(\tau)d\tau$ giving

$$-\ln\lambda(t) = \frac{(\ln 2\tau)^2}{4} - \frac{(\ln 2\tau)\ln\ln 2\tau}{2} + O(\ln 2\tau), \tag{14}$$

[5] A similar analysis applies also in the subcritical case $M < 8\pi$ where the solution converges to a self-similar one as $\tau \to \infty$, which vanishes as $t \to \infty$. In this case the operator $\mathcal{L}_a$ has strictly positive spectrum.

[6] Presently, without taking into account the nonlinearity in the equation $\partial_\tau\phi = -L_{abc}\phi + F_{abc} + N(\phi)$ [48].

while τ is related to t by

$$\begin{aligned}(T-t) &= \int_\tau^\infty \lambda(\tau')^2 d\tau' \\ &= (1/2)\left[(\ln 2\tau)^{-1} + O((\ln 2\tau)^{-2}\ln\ln 2\tau)\right] \\ &\quad\times \exp\left[-\frac{(\ln 2\tau)^2}{2} + (\ln 2\tau)(\ln\ln 2\tau) + O(\ln 2\tau)\right],\end{aligned} \tag{15}$$

where the integral was computed asymptotically in a limit $\tau \gg 1$ and T is a constant of integration determined by initial conditions. Solving the Eqs. (14) and (15) together for $t \to T$ yields the law

$$\lambda(t) = (T-t)^{\frac{1}{2}} e^{-\left|\frac{1}{2}\ln(T-t)\right|^{\frac{1}{2}}}(c_1 + o(1)), \tag{16}$$

which coincides in the leading order (up to the constant c_1) with the one obtained in [12,45,46]. The constant c_1 can be obtained only if we consider a next order correction beyond the accuracy of the Eq. (13) (see [45,46]) which is outside the scope of this paper.

The above arguments show that the spectral analysis of the linearized equation on the collapse or blowup profile is the key step in describing critical collapse or blowup laws for nonlinear evolution equations. (This also applies to stability analysis of stationary and traveling wave solutions.) Typically, this is a rather subtle affair with very few general techniques available. In this paper we develop one such a techniques for differential operators generalizing those of the form (7),

$$\mathcal{L} := -\Delta^{(4)} - \frac{8}{(1+y^2)^2} + \frac{1}{4}a^2y^2 + W_a(y), \tag{17}$$

defined on the space $L^2([0,\infty), y^3dy)$, which is the subspace of radially symmetric functions in $L^2(\mathbb{R}^4)$. Here, as indicated above, the parameter a is assumed small and positive and the potential W_a, to satisfy the bound

$$0 \le W_a(x) \le \frac{Ca}{1+y^2}, \tag{18}$$

where C is a positive constant. We assume that $W_a(x)$ is positive in order to fix the bottom of the spectrum: now $\mathcal{L} \ge 0$ (see below). Our main result is the derivation of an approximate equation for the low-lying eigenvalues of $\mathcal{L}$, which enter into the stability analysis mentioned above. Let Ψ be the digamma function, defined by $\Psi(s) = \frac{d}{ds}\ln\Gamma(s)$, where $\Gamma(s)$ is the gamma function. Define

$$0 \le \mu := 2\int_0^\infty \frac{W_a(y)}{(1+y^2)^2} y^3 dy \le Ca$$

and $K := \ln 2 - 1 - 2\gamma$, where $\gamma = -\Psi(1) = 0.577216\ldots$ is the Euler–Mascheroni constant. We have

Theorem 1. *The operator $\mathcal{L}$ is self-adjoint on $L^2([0,\infty), y^3dy)$, positive and its spectrum is discrete. For a small, the eigenvalues of $\mathcal{L}$ in the interval $[0, Ca]$, for any given $C > 0$, satisfy the equation*

$$\frac{\lambda}{\mu+a}\left[\ln\frac{1}{a} - \Psi\left(1-\frac{\lambda}{2a}\right) + K\right] = 1 + O\left(a^{1/2}\ln\frac{1}{a}\right). \tag{19}$$

As $a \to 0$, solutions of

$$\frac{\lambda}{\mu+a}\left[\ln\frac{1}{a} - \Psi\left(1-\frac{\lambda}{2a}\right) + K\right] = 1 \tag{20}$$

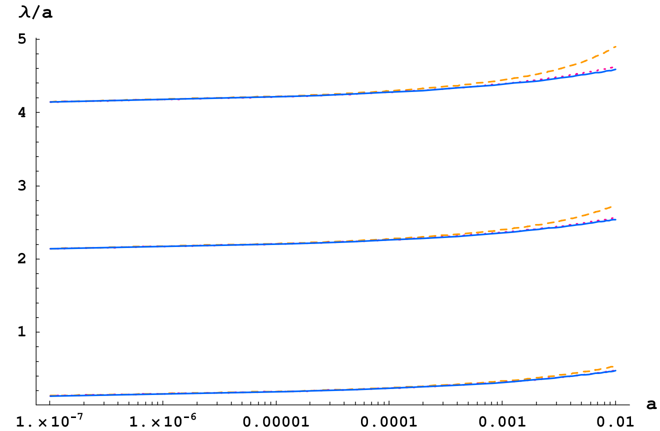

Fig. 1. The quantity λ_i/a is plotted against a for the first three eigenvalues obtained in three different ways. The solid lines are numerically computed, the dashed lines are obtained using the expressions in (21), and the dotted line (coinciding with the numerical computation for the first eigenvalue) is obtained by numerically solving the eigenvalue Eq. (20). The eigenvalue plots obtained by solving the full equation and the Eq. (20) are very close.

converge to the eigenvalues of $\mathcal{L}$. We solve this equation approximately in Section 5 to obtain

$$\lambda_n = \begin{cases} \dfrac{\mu+a}{\ln\frac{1}{a} + K + \gamma} + O\left(a\ln^{-3}\dfrac{1}{a}\right) \\ n = 0 \\ 2na + \dfrac{2a}{\ln\frac{1}{a} + K + \gamma - H_{n-1} - \frac{\mu+a}{2an}} + O\left(a\ln^{-3}\dfrac{1}{a}\right) \\ n \ge 1, \end{cases} \tag{21}$$

where $H_n := \sum_{k=1}^n 1/k$. These approximations to the eigenvalues, especially the one obtained by solving numerically (20), match remarkably well with the numerical computation of the spectrum of $\mathcal{L}$, the results of which are given in Fig. 1. The fact that the numerical solution to (20) gives much better approximation to the true eigenvalues is not surprising: the approximation (21) has the logarithmic error while the Eq. (20) is obtained with the power accuracy.

Fig. 1 compares the eigenvalue approximations obtained using (20) and (21) against the numerical computation of the first three eigenvalues of $\mathcal{L}$. We have taken $W_a = 2a/(1+y^2)$ (this gives $\mu = a$). Numerical procedure is explained in Section 6. The high precision numerical computations confirm the spectral picture we have obtained analytically.

We analyze the spectrum of $\mathcal{L}$ in the interval $[0, Ca]$ for any fixed C independent of a. This is sufficient for the stability analysis for the problem described above. However, we believe that our results are valid in a larger interval.

We have already discussed the self-adjointness of $\mathcal{L}$ on $L^2([0,\infty), y^3dy)$ and the discreteness of its spectrum. The scaling symmetry of (1) implies that the function $\eta_1(y) := 1/(1+y^2)$ is a null vector of the operator

$$\mathcal{L}_0 := -\Delta^{(4)} - \frac{8}{(1+y^2)^2}.$$

By the Perron–Frobenius argument this implies that the above operator is non-negative, $\mathcal{L} \ge 0$, and has the non-degenerate eigenvalue at 0.

As was mentioned above, Theorem 1 is proven by a rigorous version of the method of matched asymptotics (see [54]). Though this method is standard; other instances of its use in spectral problems can be found in [55–59], we however believe that our extension of this method is novel and robust and can be used for a large variety of linear differential operators arising in the linearization of nonlinear equations and hopefully can be extended to nonlinear ones as well (in this case perturbation series below should be replaced by fixed point iterations).

Indeed, like arguments outlined above, our approach is fairly general. It requires essentially only the properties listed above: the scaling symmetry and existence of a stationary solution. (In case of asymptotic motion of solitons, the scaling symmetry is replaced by translational, or more generally Galilean or Poicaré, invariance.) Indeed, the potential term $-\frac{8}{(1+y^2)^2}$ comes from linearizing the nonlinear part of the equation on the stationary solution $\chi(y)$ ($\frac{8}{(1+y^2)^2} = \frac{1}{y}\partial_y\chi(y)$), while the confining potential $\frac{1}{4}a^2y^2$ comes from the term (vector-field), $ay\partial_y$, generated by the time-dependent rescaling, and it occurs in all such problems. We do not use the explicit form of $-\frac{8}{(1+y^2)^2}$ (besides its asymptotics at $y \to \infty$ and at $y \to 0$), but the fact that, since the stationary solution, $\chi(y)$, breaks the scaling symmetry, it leads to the zero mode $\eta_1(y) := y\partial_y\chi(y)$ of the original linearization, $\mathcal{L}_0 := \gamma_0^{1/2}L_0\gamma_0^{-1/2} = -\Delta^{(4)} - \frac{8}{(1+y^2)^2}$ (after the transformation $\xi \to \gamma_0^{1/2}\xi$).

Finally, we mention the major limitation of our setup - we deal with radially symmetric solutions. Since the only stationary solution is radially symmetric (unfortunately, in contrast to biological observations), the linearized operator for the full equation commutes with rotations and therefore can be decomposed in the direct sum of radial operators. Hopefully, our analysis can be extended to each component of this sum.

This paper is organized as follows. In Section 2, using perturbation theory, we solve the eigenvalue problem

$$\mathcal{L}\phi_\lambda = \lambda\phi_\lambda \tag{22}$$

in the inner region and then proceed to find the leading order expression. We also use perturbation theory in Section 3 to solve (22) in the outer region and find the leading order behavior of this solution. In Section 4 we match the inner and outer solutions and in Section 5 we solve the Eq. (20) to obtain the solutions (21). Finally, in Section 6 we briefly discuss our numerical procedure. In Appendix we give explicit derivations of some of the expressions of Section 3, which were obtained with reference to the theory of special functions.

In what follows, we use the notation $f \lesssim g$ for $f, g \geq 0$, if there exists a positive constant C such that $f \leq Cg$ holds. If the inequality $|f| \leq C|g|$ holds then we write $f = O(g)$. We also write $f \ll g$ or $f = o(g)$ if $f/g \to 0$ as a or y approach some limit (always specified) and $f \sim g$ if the quotient converges to 1.

2. Solutions in the inner region

In what follows, $\lambda > 0$ is a spectral parameter and $0 < a \ll 1$. To simplify the expression we assume in what follows that $\lambda \leq Ca$ for some $C > 0$. Below we take $R_i = \varepsilon_i/a^{\frac{1}{2}}$ with $a \ll \varepsilon_i \ll \sqrt{\frac{a}{\lambda}\frac{1}{\ln\frac{1}{a}}}$.

The main result of this section is the following

Proposition 2. *If a is small enough, then the solution to the eigenvalue problem (22) in the inner region $[0, R_i]$ is unique, modulo an overall constant factor. For $y \in [R_o, R_i]$, $R_o \gg 1$ as $a \to 0$, this solution is given by*

$$\phi_\lambda^{in} = \frac{1}{y^2} - \frac{1}{4}\lambda\ln y^2 + \frac{1}{4}(2\lambda+\mu) + \mathcal{R}_i, \tag{23}$$

where

$$\mathcal{R}_i = O\left(\lambda\frac{\ln^2 y}{y^2} + \frac{1}{y^4} + a^2y^2 + \varepsilon_i^4 + \left[\frac{\lambda}{a}\varepsilon_i^2\ln\frac{1}{a}\right]^2\frac{1}{y^2}\right) \tag{24}$$

and the $O(f)$ signifies the bound $|O(f)| \leq Cf$ with a uniform constant C.

Proof. For $R_i > 0$, let $\|\cdot\|_i$ be the norm defined on measurable functions $f : [0, R_i] \to \mathbb{R}$ by

$$\|f\|_i := \sup_{y\leq R_i}\left|(1+y^2)f\right|$$

and let $X_i := \{f : [0, R_i] \to \mathbb{R} \mid \|f\|_i < \infty\}$ be the associated Banach space.

Eq. (22) can be written as $(\mathcal{L}_i + V_i)\phi_\lambda = 0$, where $\mathcal{L}_i$ and V_i are defined as

$$\mathcal{L}_i := -\frac{d^2}{dy^2} - \frac{3}{y}\frac{d}{dy} - \frac{8}{(1+y^2)^2} \quad \text{and}$$

$$V_i := -\lambda + \frac{1}{4}a^2y^2 + W_a(y).$$

The operator $\mathcal{L}_i$ is self-adjoint on $L^2([0,\infty), y^3dy)$ with essential spectrum $\sigma_{ess}(\mathcal{L}_i) = [0,\infty)$. It is straightforward to check that the function

$$\eta_1(y) := \frac{1}{1+y^2}$$

is a solution to the zero mode equation $\mathcal{L}_i\eta = 0$ (as mentioned in the Introduction, this is due to the scaling symmetry of (1)). Moreover, positivity of η_1 and the Perron–Frobenius theorem imply that $0 = \inf\sigma(\mathcal{L}_i)$, and hence $\mathcal{L}_i$ is a positive operator, but 0 is not an eigenvalue. The function η_1 is not an eigenvector of $\mathcal{L}_i$ since it does not lie in $L^2([0,\infty), y^3dy)$; we call it a resonance eigenvector. Using that the Wronskian of $\mathcal{L}_i$ is $1/y^3$, we find a second solution of $\mathcal{L}_i\eta = 0$:

$$\eta_2 := \eta_1\left(\frac{1}{2}y^2 + \ln y^2 - \frac{1}{2}y^{-2}\right).$$

This vector is independent of η_1, also does not lie in $L^2([0,\infty), y^3dy)$ and has a singularity at $y = 0$. Note that $\eta \in X_i$.

Using variation of parameters we find that the general solution to $\mathcal{L}_i\xi = f$ is $\xi(y) = (\mathcal{L}_i^{-1}f)(y) + C_1\eta_1(y) + C_2\eta_2(y)$, where C_1 and C_2 are arbitrary constants and

$$(\mathcal{L}_i^{-1}f)(y) := \eta_1(y)\int_0^y \eta_2(x)f(x)x^3dx - \eta_2(y)\int_0^y \eta_1(x)f(x)x^3dx. \tag{25}$$

Lemma 3. *Say that $a \ll \varepsilon_i \ll 1$. For a small enough, Eq. (22) has a unique, modulo an overall constant factor, solution on $[0, R_i]$ of the form $\phi_\lambda^{in} = \eta_1 + \xi$ with $\xi \in X_i$ and*

$$\xi = \sum_{n=1}^{\infty}(-\mathcal{L}_i^{-1}V_i)^n\eta_1. \tag{26}$$

The convergence is in X_i and $\|\mathcal{L}_i^{-1}V_i\|_{X_i\to X_i} \lesssim (\frac{\lambda}{a}+1)\varepsilon_i^2\ln\frac{1}{a}$.

Remark 1. In fact, we can show convergence of the series in an appropriate norm without the condition on the parameter a and the range of y.

Proof. Recall that $\mathcal{L} = \mathcal{L}_i + V_i$. Substituting $\phi_\lambda^{in} = \eta_1 + \xi$ into (22) and using $\mathcal{L}_i\eta_1 = 0$ and the general solution of $\mathcal{L}_i\xi = f$ found above, we obtain that $\xi = -\mathcal{L}_i^{-1}(V_i\xi + V_i\eta) + C_1\eta_1 + C_2\eta_2$. The term $C_1\eta_1$ leads to change of an overall factor multiplying ϕ_λ^{in} and therefore it can be dropped. Next, if $\xi \in X_i$, then so is $\mathcal{L}_i^{-1}V_i\xi$ (see below). Since $\eta_2 \notin X_i$, we must, therefore, take $C_2 = 0$, otherwise we would have a contradiction. Finally, we rearrange the resulting equation to obtain that

$$(I + \mathcal{L}_i^{-1}V_i)\xi = -\mathcal{L}_i^{-1}V_i\eta_1. \tag{27}$$

As (25) shows, (27) is a Volterra equation and therefore the operator on the r.h.s. has an inverse and this inverse can be expanded in the standard perturbation series. We can invert the operator on the left hand side on the space X_i, provided $\|\mathcal{L}_i^{-1}V_i\|_{X_i\to X_i} < 1$. To show the latter property, we compute that

$$\|\mathcal{L}_i^{-1}V_if\|_i \le \left\{\sup_{y\le R_i}|\rho(y)\eta_1(y)|\int_0^y \frac{|\eta_2(z)V_i(z)|}{\rho(z)}z^3dz + \sup_{y\le R_i}|\rho(y)\eta_2(y)|\int_0^y \frac{|\eta_1(z)V_i(z)|}{\rho(z)}z^3dz\right\}\|f\|_i, \quad (28)$$

where $\rho(y) = (1+y^2)$ and $f \in X_i$. Substituting the expressions for ρ, η_1, η_2 and V_i into the first term and using (18) gives that

$$\begin{aligned}|\rho\eta_1|\int_0^y \frac{|\eta_2V_i|}{\rho}(z)\,z^3dz &= \int_0^y \frac{1}{(1+z^2)^2}\left|\frac{1}{2}z^2+\ln z^2-\frac{1}{2}\frac{1}{z^2}\right| \\ &\quad\times\left|-\lambda+\frac{a^2}{4}z^2+W_a(z)\right|z^3dz \\ &\lesssim \int_0^y\left(\lambda+a^2z^2+\frac{a}{1+z^2}\right)z\,dz. \quad (29)\end{aligned}$$

This gives that $|\rho\eta_1|\int_0^y \frac{|\eta_2V_i|}{\rho}(z)\,z^3dz \lesssim a^2y^4 + a\ln(1+y^2)+\lambda y^2$, and hence

$$\sup_{[0,R_i]}|\rho\eta_1|\int_0^y \frac{|\eta_2V_i|}{\rho}(z)\,z^3dz \lesssim a^2R_i^4 + a\ln R_i^2 + \lambda R_i^2. \quad (30)$$

Similarly, we compute that

$$\begin{aligned}|\rho\eta_2|\int_0^y \frac{|\eta_1V_i|}{\rho}(z)\,z^3dz \\ \lesssim \left(y^2+\frac{1}{y^2}\right)\left(ay^4+\lambda y^4(1+\ln(1+y^2))+a^2y^6\right)(1+y^2)^{-2}\end{aligned}$$

and hence,

$$\sup_{[0,R_i]}|\rho\eta_2|\int_0^y \frac{|\eta_1V_i|}{\rho}(z)\,z^3dz \lesssim a^2R_i^4 + \lambda R_i^2\ln R_i + (a+\lambda)R_i^2. \quad (31)$$

Substituting the definition $R_i := \varepsilon_i/a^{1/2}$ into (30) and (31), using $1 \gg \varepsilon_i \gg a$ to simplify $\ln\frac{\varepsilon_i^2}{a}$ to $\ln\frac{1}{a}$ and then using the results in (28) gives that $\|\mathcal{L}_i^{-1}V_i\|_{X_i\to X_i} \lesssim \varepsilon_i^2\frac{\lambda}{a}\ln\frac{1}{a}$ and hence a can be taken small enough so that $\|\mathcal{L}_i^{-1}V_i\|_{X_i\to X_i} < 1$. Now inverting the operator on the l.h.s. of (27) and expanding the inverse into the Neumann series completes the proof. □

The expression (23) for ϕ_λ^{in} is obtained as follows. Due to Lemma 3 and since $\|\eta_1\|_i = O(1)$, if a is small enough (and $\varepsilon_i \ll 1$), then

$$\phi_\lambda = \eta_1 - \mathcal{L}_i^{-1}V_i\eta_1 + O_{X_i}\left(\varepsilon_i^4 + \left[\frac{\lambda}{a}\varepsilon_i^2\ln\frac{1}{a}\right]^2\right), \quad (32)$$

where the $O_{X_i}(\epsilon)$ stands for a function bounded as $\|O_{X_i}(\epsilon)\|_{X_i} \lesssim \epsilon$. Since $|f| \le \frac{1}{1+y^2}\|f\|_i$, the latter is the same as $|O_{X_i}(\epsilon)| \lesssim \epsilon(1+y^2)^{-1}$. Substituting the large y expansions $\eta_1 = y^{-2} + O\left(y^{-4}\right)$ and $\eta_2 = 1/2 + O\left(\ln(y)/y^2\right)$ into the expression for $\mathcal{L}_i^{-1}V_i\eta_1$ (see (25)) and keeping only leading terms needed in the region $[R_o, R_i]$ gives the expression

$$\mathcal{L}_i^{-1}V_i\eta_1 = -\frac{2\lambda+\mu}{4} + \frac{\lambda}{4}\ln y^2 + O\left(\lambda\frac{\ln^2 y}{y^2} + (a+|\mu|)\frac{\ln y}{y^2} + a^2y^2\right).$$

Substituting this expression into (32), with η_1 replaced with its large y expansion and using that

$$\left|O_{X_i}\left(\varepsilon_i^4 + \left[\frac{\lambda}{a}\varepsilon_i^2\ln\frac{1}{a}\right]^2\right)\right| \lesssim \left(\varepsilon_i^4 + \left[\frac{\lambda}{a}\varepsilon_i^2\ln\frac{1}{a}\right]^2\right)(1+y^2)^{-1},$$

gives (23). □

3. Solutions in the outer region

In the following discussion we take $R_o := \varepsilon_o/a^{\frac{1}{2}}$ with $\frac{a^{\frac{1}{2}}}{\ln^{\frac{1}{4}}\frac{1}{a}} \ll \varepsilon_o \ll 1$ as $a \to 0$. The main result of this section is the following

Proposition 4. *On $[R_o,\infty)$ (22) has a unique solution ϕ_λ^{out}, which, for $y \in [R_o, R_i]$, takes the form*

$$\phi_\lambda^{out} = \ln y^2 - \ln\frac{1}{a} - \ln 2 - 1 + 2\gamma + \Psi\left(1-\frac{\lambda}{2a}\right) - \frac{4}{\lambda}\frac{1}{y^2} + \frac{a}{\lambda} + \mathcal{R}_o, \quad (33)$$

where

$$\mathcal{R}_o = O\left(ay^2|\ln(ay^2)| + \frac{|\ln(aR_o^2)|}{R_o^2}(ay^2)^{\frac{\lambda}{2a}-1}\right). \quad (34)$$

Proof. For a, R_o and on measurable functions $f : [R_o,\infty) \to \mathbb{R}$, we define the norm

$$\|f\|_o := \sup_{y\ge R_o}\left|ay^2(ay^2+1)^{-\frac{\lambda}{2a}}e^{\frac{a}{4}y^2}f\right|.$$

Let X_o be the corresponding Banach space of functions defined on $[R_o, 0)$ with finite norm $\|f\|_o$.

We write (22) as $(\mathcal{L}_o + V_o)\phi_\lambda = 0$, where

$$\mathcal{L}_o := -\frac{d^2}{dy^2} - \frac{3}{y}\frac{d}{dy} + V(ay) - \lambda \quad \text{and}$$

$$V_o := U(y) + W_a(y). \quad (35)$$

(Recall that $V(ay) = \frac{1}{4}a^2y^2$ and $U(y) = -\frac{8}{(1+y^2)^2}$.) In the outer region $y \ge R_o$, we treat V_o as a small potential. The operator $\mathcal{L}_o$ is self-adjoint on $L^2([0,\infty), y^3dy)$.

In what follows, the relation $f \sim g$ as $y \to \infty$ means that f/g converges to a constant (which might depend only on $\frac{\lambda}{2a}$) as $y \to \infty$ and the notation $f \approx g$ as $y \to 0$ means that f/g converges to 1 as $y \to 0$.

We show in Appendix A, Proposition 7, that the equation $\mathcal{L}_o\phi = 0$ which is the eigenvalue equation for the spherically symmetric harmonic oscillator in $D = 4$, has two linearly independent solutions, ϕ_0 and ϕ_1, satisfying the estimates

$$|\phi_0| \lesssim \left|\Gamma\left(-\frac{\lambda}{2a}\right)\right|(ay^2)^{-1}(ay^2+1)^{\frac{\lambda}{2a}}, e^{-\frac{a}{4}y^2} \quad (36)$$

and

$$|\phi_1| \lesssim \frac{1}{\left|\Gamma\left(1-\frac{\lambda}{2a}\right)\right|}(ay^2+1)^{-\frac{\lambda}{2a}-1}e^{\frac{a}{4}y^2}, \quad (37)$$

and having the Wronskian

$$W(\phi_0,\phi_1) = -\frac{8}{\lambda y^3}. \quad (38)$$

Hence $\phi_0 \in X_o$. Using variation of parameters and the Wronskian (38), we find the general solution to $\mathcal{L}_o\xi = f$ as $\xi = \mathcal{L}_o^{-1}f + C_1\phi_0 +$

$C_2\phi_1$, where C_1 and C_2 are arbitrary constants and

$$\mathcal{L}_o^{-1}f := -\frac{\lambda}{8}\left(\phi_0\int_y^\infty \phi_1 f\, y^3 dy - \phi_1\int_y^\infty \phi_0 f\, y^3 dy\right).$$

Lemma 5. *Say that* $\frac{\lambda}{\varepsilon_o^2}|\ln(\varepsilon_o^2)| \ll 1$. *If the parameter* a *is small enough, the Eq.* (22) *has a unique, modulo and overall constant factor, solution on* $[R_o,\infty)$ *of the form* $\phi_0+\xi$, *with* $\xi\in X_o$ *and*

$$\xi = \sum_{n=1}^{\infty}(-\mathcal{L}_o^{-1}V_o)^n\phi_0. \tag{39}$$

The series converges absolutely in X_o *and*

$$\|\mathcal{L}_o^{-1}V_o\|_{X_o\to X_o} \lesssim \frac{1}{R_o^2}|\ln(aR_o^2)|. \tag{40}$$

Proof. Substituting $\phi_\lambda^{out}=\phi_0+\xi$ into $(\mathcal{L}_o+V_o)\phi=0$ and using that $\mathcal{L}_o\phi_0=0$, we obtain $\mathcal{L}_o\xi+V_o\xi=-V_o\phi_o$. Now, using the form of the general solution to $\mathcal{L}_o\xi=f$ found above and that $\phi_1\notin X_o$ gives that $(I+\mathcal{L}_o^{-1}V_o)\xi=-\mathcal{L}_o^{-1}V_o\phi_0$. We note that the choice of the constants $C_1=0$ and $C_2=0$ follows from similar arguments as in Lemma 3. The operator on the left hand side can be inverted using the Neumann series if $\|\mathcal{L}_o^{-1}V_o\|_{X_o\to X_o}<1$, and hence we estimate $\left\|\mathcal{L}_o^{-1}V_o f\right\|_o$ for $f\in X_o$:

$$\left\|\mathcal{L}_o^{-1}V_o f\right\|_o \le \frac{\lambda}{8}\sup_{y\ge R_o}\left\{|\rho\phi_0|\int_y^\infty\left|\frac{\phi_1 V_o}{\rho}\right| z^3dz \right.$$
$$\left. +|\rho\phi_1|\int_y^\infty\left|\frac{\phi_0V_o}{\rho}\right| z^3dz\right\}\|f\|_o, \tag{41}$$

where $\rho(y):=ay^2(ay^2+1)^{-\frac{\lambda}{2a}}e^{\frac{a}{4}y^2}$. For $y\ge R_o$ and R_o large, we have, using (18) and $a^{\frac12}R_o\ll 1$, that

$$|V_o(y)|\lesssim\left(a+\frac{1}{R_o^2}\right)\frac{1}{y^2}\lesssim\frac{1}{R_o^2}\frac{1}{y^2}.$$

Using this estimate and (36) and (37) in (41) yields

$$\left\|\mathcal{L}_o^{-1}V_o f\right\|_o \lesssim \frac{\lambda}{R_o^2}\sup_{y\ge R_o}\left\{\int_y^\infty \rho_0\rho_1\, zdz\right.$$
$$\left.+\rho_0^{-1}\rho_1 e^{\frac{a}{2}y^2}\int_y^\infty\rho_0^2\, e^{-\frac{a}{2}z^2}zdz\right\}\|f\|_o, \tag{42}$$

where $\rho_0=\rho_0(\frac{ay^2}{2})$ and $\rho_1=\rho_1(\frac{ay^2}{2})$ are the prefactors on the r.h.s. of (36) and (37), respectively. Changing the variables of integration as $t=\frac{ay^2}{2}$ and using that $aR_o^2=\varepsilon_o^2\ll 1$, we obtain (40).

Hence, for a is small enough so that the r.h.s. of (40) <1, we have that $\|\mathcal{L}_o^{-1}V_o\|_{X_o\to X_o}<1$, and therefore the series (39) converges absolutely, which completes the proof. □

Lemma 5 shows that in $[R_o,\infty)$, ϕ_0 solves (22) in leading order in a:

$$\phi_\lambda^{out}=\phi_0+O_{X_o}\left(\frac{1}{R_o^2}|\ln(aR_o^2)|\right). \tag{43}$$

Next, we show in Appendix that ϕ_0 has the following asymptotics

$$\phi_0=\ln(ay^2)-\ln 2-1+2\gamma+\Psi\left(1-\frac{\lambda}{2a}\right)-\frac{4}{\lambda}\frac{1}{y^2}$$
$$+\frac{a}{\lambda}+O\left(ay^2\ln(ay^2)\right), \tag{44}$$

where, recall, Ψ is the digamma function and $\gamma=-\Psi(1)=0.577216\ldots$ is the Euler–Mascheroni constant.

Expressions (43) and (44) and the observation that $|f|\le(ay^2)^{-1}(ay^2+1)^{\frac{\lambda}{2a}}e^{-\frac{a}{4}y^2}\|f\|_o$ give (33)–(34). □

4. Matching and eigenvalues

In this section we prove the main result stated in the Introduction. We have two expressions, the inner and outer solutions, (23) and (33), which solve $\mathcal{L}\phi_\lambda=\lambda\phi_\lambda$ in the common region $[R_0,R_i]$. The inner and outer solutions, (23) and (33), should be equal, up to a constant multiple, in the common region $[R_0,R_i]$. Hence we require that

$$-\frac14\lambda\ln y^2+\frac{1}{y^2}+\frac14(2\lambda+\mu)+\mathcal{R}_i$$
$$=C\left(\ln y^2-\frac{4}{\lambda}\frac{1}{y^2}-\ln\frac1a-\ln 2-1+2\gamma+\frac{a}{\lambda}\right.$$
$$\left.+\Psi\left(1-\frac{\lambda}{2a}\right)+\mathcal{R}_o\right) \tag{45}$$

for $y\in[R_o,R_i]$. Here $\mathcal{R}_i$ and $\mathcal{R}_o$ are given in (24) and (34).

Note that the remainders in (23) and (33) are much smaller than the corresponding leading terms, i. e. $|\mathcal{R}_i|\ll a$ and $|\mathcal{R}_o|\ll 1$, if

$$y\gg\ln\frac1a\sqrt{\ln R_o}\varepsilon_iR_i\quad\text{and}\quad ay^2\ll 1, \tag{46}$$

respectively. We assume that $a\to 0$ and that (46) holds. Equating the leading terms in Eq. (45), i.e. the terms which multiples of $\ln y^2$, gives the equation

$$C=-\frac{\lambda}{4}+R_C,$$

with $|R_C|\lesssim\inf\frac{a|\mathcal{R}_o|+|\mathcal{R}_i|}{\ln y}$, where inf is taken over $R_o\le y\le R_i$ satisfying the condition (46), and therefore $R_C=O\left(a^{3/2}\right)$. (Here and below we take $\varepsilon_i\sim\varepsilon_o\sim a^{1/4}$.) Similarly, equating the constant terms in (45) and substituting the above expression for C gives that

$$2\lambda+\mu$$
$$=\lambda\left(\ln\frac1a+\ln 2+1-2\gamma-\frac{a}{\lambda}-\Psi\left(1-\frac{\lambda}{2a}\right)\right)+R,$$

where the higher order term R is bounded as $|R|\lesssim\inf(a|\mathcal{R}_o|+|\mathcal{R}_i|)$ and therefore satisfies

$$R=O\left(a^{3/2}\ln\left(\frac1a\right)\right).$$

Rearranging the above equation, assuming that $\mu+a\ne 0$, gives that

$$1=\frac{\lambda}{\mu+a}\left[\ln\frac1a-\Psi\left(1-\frac{\lambda}{2a}\right)+K\right]+\frac{R}{\mu+a},$$

where, recall, $K:=\ln 2-1-2\gamma$. This is the Eq. (19). This proves Theorem 1. □

5. Solution of (20)

Proposition 6. *The set of solutions to* (20) *as* $a\to 0$ *is* $\{\lambda_n\}_{n=0}^\infty$, *where* λ_n *is given by* (21).

Proof. The term $\ln\frac1a$ on the left hand side of (20) is unbounded as $a\to 0$ whereas the right hand side is bounded. Thus, there are two possibilities: either $\lambda/(\mu+a)\ll 1$ as $a\to 0$ or $|\lambda/(a+\mu)|\ge C>0$ and there is cancelation between $\ln\frac1a$ and $\Psi(1-\lambda/2a)$.

We begin with the first case. If $\lambda/(\mu+a) \ll 1$ as $a \to 0$, then, since $\mu \lesssim a$, $\lambda/2a \sim \lambda/(\mu+a)$ as $a \to 0$. We use this and the fact that $\Psi(1) = -\gamma$ and $\Psi(1+\delta) = \Psi(1) + O(\delta)$ to write (20) as

$$\frac{\lambda}{\mu+a}\left[\ln\frac{1}{a} + K + \gamma + O\left(\frac{\lambda\sqrt{a}}{\mu+a}\right)\right] = 1. \tag{47}$$

This equation immediately gives the rough estimate that $\lambda/(\mu+a) = O\left(\ln^{-1}\frac{1}{a}\right)$. Substituting this estimate into the $O(\cdot)$ term in (47), then solving the resulting equation for λ gives that

$$\lambda = \frac{\mu+a}{\ln\frac{1}{a}+K+\gamma}\,\frac{1}{1+O\left(\ln^{-2}\frac{1}{a}\right)}.$$

Further simplification of the right hand side gives the $n = 0$ expression in (21).

If $|\lambda/(\mu+a)| \geq C > 0$, then there must be cancelation between $\ln\frac{1}{a}$ and $\Psi\left(1-\frac{\lambda}{2a}\right)$ in (20) as already mentioned above. The digamma function $\Psi(x)$ has poles at $x = -n$ for integers $n \geq 0$ and hence, for cancelation to occur, $\lambda/2a$ must have the form

$$\frac{\lambda}{2a} = 1 + n + \delta, \tag{48}$$

where $\delta \ll 1$ as $a \to 0$. Substituting this form of $\lambda/2a$ into (19) gives the equation

$$(1+n+\delta)\left(\ln\frac{1}{a} - \Psi(-n-\delta) + K\right) = \frac{\mu+a}{2a}. \tag{49}$$

We extract the singular behavior of $\Psi(-n-\delta)$ using the identity

$$\Psi(-n-\epsilon) = \Psi(1-\delta) + \sum_{k=0}^{n}\frac{1}{k+\delta}. \tag{50}$$

If $k \geq 1$ and $\delta < 1/2$, then $1/(k+\delta) \lesssim 1/k + \delta/k^2$. Using this bound and the fact that $\Psi(1-\delta) = -\gamma + O(\delta)$ in (50) yields that

$$\Psi(-n-\delta) = \frac{1}{\delta} - \gamma + \sum_{k=1}^{n}\frac{1}{k} + O(\delta).$$

Substituting the right hand side for $\Psi(-n+\delta)$ in (49) gives the equation

$$(1+n+\delta)\left(\ln\frac{1}{a} - \frac{1}{\delta} + K + \gamma - \sum_{k=1}^{n}\frac{1}{k} + O(\delta)\right) = \frac{\mu+a}{2a}.$$

As before, a rough estimate of δ is $\ln^{-1}\frac{1}{a}$. Using this to invert $(1+n+\delta)$ gives that

$$\ln\frac{1}{a} - \frac{1}{\delta} + K + \gamma - \sum_{k=1}^{n}\frac{1}{k} + O\left(\ln^{-1}\frac{1}{a}\right) = \frac{\mu+a}{2a(1+n)}\,\frac{1}{1+O\left(\ln^{-1}\frac{1}{a}\right)}$$

and hence, solving this equation for δ, we find that

$$\delta = \frac{1}{\ln\frac{1}{a} + K + \gamma - \sum_{k=1}^{n}\frac{1}{k} - \frac{\mu+a}{2a(n+1)}} + O\left(\ln^{-3}\frac{1}{a}\right).$$

Substituting this expression into (48) gives the eigenvalue approximation (21) for $n \geq 1$ (with n replaced by $n-1$ above) and completes the proof. □

6. Numerical calculation of spectrum

To determine eigenvalues and eigenfunctions numerically we used a version of the shooting method: we numerically solved the eigenvalue Eq. (20) with initial conditions $\phi_\lambda(y=0) = 1$ and $\frac{d}{dy}\phi_\lambda(y=0) = 0$ for each value of λ. For general λ, the solution at $ay^2 \gg 1$ is a linear combination (54), which grows exponentially as given by (57). We used Newton's method to find values λ for which $c_1 = 0$ (i.e. removing exponentially growing terms at infinity) in (54). The stopping criterion for Newton's method was to have both $\phi_\lambda(y)$ and $\frac{d}{dy}\phi_\lambda(y)$ to decay exponentially for large y. Note that vanishing of $\phi_\lambda(y)$ for $y \to \infty$ is not sufficient because it would not exclude a spurious solution when $c_1\phi_0(y) + c_2\phi_1(y) = 0$ at one point only. We controlled numerical precision by the matching of numerical solution to the asymptotics (57).

Fig. 1 shows the first three eigenvalues as functions of a obtained in three different ways. The solid lines are numerically computed from the shooting method, the dashed lines are obtained using the expressions in (21), and the dash–dot lines (almost visually indistinguishable from solid lines) are obtained by (20). It is seen that accuracy of numerical solutions compared with approximate analytical results is very high.

7. Conclusions

We summarize the main results of this paper:

- We found low-lying spectrum for a class of operators which appear in the linearization of the simplified critical Keller–Segel system around the one-parameter family of stationary solutions. These operators have one negative and one near zero eigenvalue and as a result – as discussed in the Introduction – the blowup asymptotics will be governed by a two-parameter deformation of the static solution. The eigenfunctions corresponding to these eigenvalues suggest the construction of such deformations which, together with ensuing results, are outlined in the Introduction.
- We constructed a rigorous and robust version of the method of matched asymptotics. We believe it can be used with a large variety of linear differential operators arising in the linearization of nonlinear equations and hopefully can be extended to nonlinear ones as well (in this case perturbation series, (26) and (39), should be replaced by fixed point iterations).

There are two main limitations of our setup: we deal with radially symmetric solutions and with adiabatic approximation ignoring evolution of chemical concentration. Hence we conclude by emphasizing the desirability of two further extensions of our analysis by considering

- non-radially symmetric initial conditions;
- the full Keller–Segel model (without the adiabatic approximation).

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Appendix. Solutions of $\mathcal{L}_0\phi = 0$

In this appendix we derive using well-known properties of the confluent hypergeometric functions, some properties of solutions of equation $\mathcal{L}_o\phi = 0$ which were used in Section 3.

Proposition 7. *If $\lambda/2a \neq$ is a positive integer, then there are two independent solutions, ϕ_0 and ϕ_1, of the equation $\mathcal{L}_0\phi = 0$, satisfying the bounds (36) and (37), and having the Wronskian (38) and the expansion (44).*

Proof. The equation $\mathcal{L}_o\phi = 0$ is the eigenvalue equation for the spherically symmetric harmonic oscillator in $D = 4$:

$$\mathcal{L}_o\phi = \left[-\frac{1}{y^3}\partial_y y^3\partial_y + \frac{a^2}{4}y^2 - \lambda\right]\phi = 0. \tag{51}$$

Changing in this equation both dependent and independent variables,

$$\phi(y) = e^{-ay^2/4}\chi(z), \quad z = \frac{ay^2}{2}, \tag{52}$$

we obtain the Kummer's (or a confluent hypergeometric) differential equation [60]

$$z\chi'' + (2-z)\chi' + \left(\frac{\lambda}{2a} - 1\right)\chi = 0. \tag{53}$$

Assuming that $\lambda/2a \neq$ is a positive integer, the latter equation has two linearly independent solutions

$$\chi_0 = U\left(1 - \frac{\lambda}{2a}, 2, z\right), \qquad \chi_1 = {}_1F_1\left(1 - \frac{\lambda}{2a}, 2, z\right), \tag{54}$$

where $U(a, b, z)$ and ${}_1F_1(a, b, z)$ are the confluent hypergeometric functions of the second kind and the first kind, respectively [60]. They are given by the following expressions (see the equations (13.1.2), (13.1.6) in [60] and the equation (13) of the Section 6.7.1 of [61]):

$${}_1F_1(a, b, z) = \sum_{r=0}^{\infty} \frac{(a)_r z^r}{(b)_r r!},$$

$$\begin{aligned} &U(a, n+1, z) \\ &= \frac{(-1)^{n+1}}{n!\Gamma(a-n)}\Bigg[{}_1F_1(a, n+1, z)\ln z + \sum_{r=0}^{\infty}\frac{(a)_r z^r}{(n+1)_r r!} \\ &\quad \times \{\Psi(a+r) - \Psi(1+r) - \Psi(1+n+r)\}\Bigg] \\ &\quad + \frac{(n-1)!}{\Gamma(a)}\sum_{r=0}^{n-1}\frac{(a-n)_r}{(1-n)_r}\frac{z^{r-n}}{r!}, \end{aligned} \tag{55}$$

where $(a)_j = a(a+1)(a+2)\dots(a+j-1)$, $(a)_0 = 1, n = 0, 1, 2, \dots$, and the principal branch of $\ln z$ is assumed to be chosen by setting $-\pi < \arg z \leq \pi$.

The Eqs. (52) and (54) give two linearly independent solutions ϕ_0 and ϕ_1 of $\mathcal{L}_o\phi = 0$ on $[0, \infty)$:

$$\begin{aligned} \phi_0(y) &= \Gamma\left(-\frac{\lambda}{2a}\right)U\left(-\frac{\lambda}{2a} + 1, 2, \frac{ay^2}{2}\right)e^{-ay^2/4}, \\ \phi_1(y) &= {}_1F_1\left(-\frac{\lambda}{2a} + 1, 2, \frac{ay^2}{2}\right)e^{-ay^2/4}, \end{aligned} \tag{56}$$

where, using that ϕ_0 and ϕ_1 are defined up to arbitrary constants, we added, for convenience, the factor $\Gamma(-\frac{\lambda}{2a})$. (Without that factor the Eq. (44) would have a factor $1/\Gamma(-\frac{\lambda}{2a})$ in all terms.)

Asymptotic expansions of both confluent hypergeometric functions in (54) for $ay^2 \to \infty$ are given by (see the equations (13.1.4) and (13.1.8) of Ref. [60])

$$\begin{aligned} \phi_0(y) &= \Gamma\left(-\frac{\lambda}{2a}\right)\left(\frac{ay^2}{2}\right)^{\frac{\lambda}{2a}-1} e^{-ay^2/4}\left[1 + O\left(\frac{1}{ay^2}\right)\right], \\ \phi_1(y) &= \frac{1}{\Gamma\left(1 - \frac{\lambda}{2a}\right)}\left(\frac{ay^2}{2}\right)^{-\frac{\lambda}{2a}-1} e^{ay^2/4}\left[1 + O\left(\frac{1}{ay^2}\right)\right]. \end{aligned} \tag{57}$$

It follows from Eq. (57) that out of these two functions only ϕ_0 decays at infinity (with correct asymptotic $\phi_0 \propto y^{\frac{\lambda}{a}-2}e^{-\frac{a}{4}y^2}$).

The bounds (36) and (37) are proven similarly, so we prove only (36). Recall the definition of χ_0 in (54). (55) implies that for a constant C_1, which might depend only on $\lambda/2a$, there is a point $z_0 \geq 1$ s.t. $|\chi_0(z)| \leq C_1 z^{\frac{\lambda}{2a}-1}$ for all $z \geq z_0$. Since χ_0 depends only on z and $\lambda/2a$, there is a constant C_2, which might depend only on $\lambda/2a$, s.t. $|\chi_0(z)| \leq C_2 z^{\frac{\lambda}{2a}-1}$ for all $z \geq 1$.

Now, (55) implies that $|\chi_0(z)| \leq C_3/z$ for all $0 \leq z \leq 1$ for some absolute constant C_3. Combining the above inequalities gives the bound (36).

Now we compute the Wronskian, $W(\phi_0, \phi_1) := \phi_0\partial_y\phi_1 - \partial_y\phi_0\phi_1$, of the two solutions found above. As usual, using the equation $\mathcal{L}_0\phi = 0$, we derive the first order equation, $W' = -\frac{3}{y}W$, for $W(\phi_0, \phi_1)$. Solving this equation, we obtain $W(\phi_0, \phi_1) = \frac{C}{y^3}$, with a real constant C. To find this constant we compute W as $y \to \infty$, using the Eqs. (57) and the fact $\Gamma(z+1) = z\Gamma(z)$ (one can also find C using the expansion (55)). This gives $C = -\frac{8}{\lambda}$ and (38).

Finally, we prove (44) for ϕ_0. To this end we study the behavior of the solution for $y \ll 1/a^{\frac{1}{2}}\ln^{\frac{1}{2}}\frac{1}{a}$, $a \ll 1$, or equivalently, $z \ll 1/\ln\frac{1}{a}$. In the small z region,

$$\chi_\lambda(z) = \ln\frac{z}{A} - \frac{2a}{\lambda z} + O(z)$$

and $e^{-\frac{1}{2}z} = 1 - \frac{1}{2}z + O(z^2)$. Computing the product of the small z expansions of χ_λ and $e^{-\frac{z}{2}}$, and replacing z with $\frac{ay^2}{2}$ in the result gives the expression (44).

Now let $y \in (0, R_i]$. Using the expansion (55) for $n = 1$ and (52) we obtain (44). □

References

[1] I. Rodnianski, J. Sternbenz, On the formation of singularities in the critical $O(3)\sigma$-model, Ann. of Math. (2) 172 (1) (2010) 187–242.
[2] P. Rafael, I. Rodnianski, Stable blow up dynamics for the critical co-rotational wave map and equivariant Yang–Mills problems, Publi. I.H.E.S. (2010), in press, arXiv:0911.0692.
[3] J. Krieger, W. Schlag, D. Tataru, Renormalization and blow up for charge one equivariant critical wave maps, Invent. Math. 171 (3) (2008) 543–615.
[4] J. Krieger, W. Schlag, D. Tataru, Renormalization and blow up for critical Yang–Mills problem, Advances in Mathematics 221 (5) (2009) 1445–1521.
[5] Yu.N. Ovchinnikov, I.M. Sigal, On collapse of wave maps, Physica D 240 (2011) 1311–1324.
[6] P. Bizoń, Yu.N. Ovchinnikov, I.M. Sigal, Collapse of an instanton, Nonlinearity 17 (4) (2004) 1179–1191.
[7] F. Merle, P. Rafael, Blow up of the critical norm for some radial $L2$ super critical nonlinear Schrödinger equations, Amer. J. Math. 130 (2008) 945.
[8] F. Merle, H. Zaag, Blow-up behavior outside the origin for a semilinear wave equation in the radial case, Bull. Sci. Math. 135 (4) (2011) 353–373.
[9] S. Dejak, Z. Gang, I.M. Sigal, S. Wang, Blowup dynamics in nonlinear heat equations, Adv. Appl. Math. 40 (2008) 433.
[10] C. Sulem, P.L. Sulem, Nonlinear Schrödinger Equation, Springer, 1999.
[11] Michael P. Brenner, Leonid S. Levitov, Elena O. Budrene, Physical mechanisms for chemotactic pattern formation by bacteria, Biophys. J. 74 (4) (1998) 1677–1693.
[12] J.J.L. Velázquez, Stability of some mechanisms of chemotactic aggregation, SIAM J. Appl. Math. 62 (5) (2002) 1581–1633 (electronic).
[13] J.J.L. Velázquez, SIAM J. Appl. Math. 64 (4) (2004) 1198–1248.

[14] A. Blanchet, E. Carlen, J.A. Carrillo, Functional inequalities, thick tails and asymptotics for the critical mass Patlak–Keller–Segel model, J. Funct. Anal. 262 (5) (2012) 2142–2230.
[15] A. Blanchet, J.A. Carrillo, P. Laurençot, Critical mass for a Patlak–Keller–Segel model with degenerate diffusion in higher dimensions, Calc. Var. Partial Differential Equations 35 (2) (2009) 133–168.
[16] A. Blanchet, J.A. Carrillo, N. Masmoudi, Infinite time aggregation for the critical Patlak–Keller–Segel model in $\mathbb{R}^2$, Comm. Pure Appl. Math. 61 (10) (2008) 1449–1481.
[17] A. Blanchet, J. Dolbeault, M. Escobedo, J. Fernandez, Asymptotic behaviour for small mass in the two-dimensional parabolic–elliptic Keller–Segel model, J. Math. Anal. Appl. 361 (2) (2010) 533–542.
[18] A. Blanchet, J. Dolbeault, B. Perthame, Two-dimensional Keller–Segel model: optimal critical mass and qualitative properties of the solutions, Electron. J. Differential Equations (44) (2006) 32.
[19] A.L. Bertozzi, J.A. Carillo, Th. Laurent, Blow-up in multidimensional aggregation equations with mildly singular interaction kernels, Nonlinearity 22 (2009) 683–710.
[20] Michael P. Brenner, Peter Constantin, Leo P. Kadanoff, Alain Schenkel, Shankar C. Venkataramani, Diffusion, attraction and collapse, Nonlinearity 12 (4) (1999) 1071–1098.
[21] T. Nagai, Global existence and blowup of solutions to a chemotaxis system, in: Proceedings of the Third World Congress of Nonlinear Analysts, Part 2, Catania, 2000, vol. 47, 2001, pp. 777–787.
[22] Dirk Horstmann, From 1970 until present: the Keller–Segel model in chemotaxis and its consequences. I, Jahresber. Deutsch. Math.-Verein. 105 (3) (2003) 103–165.
[23] Dirk Horstmann, From 1970 until present: the Keller–Segel model in chemotaxis and its consequences. II, Jahresber. Deutsch. Math.-Verein. 106 (2) (2004) 51–69.
[24] T. Hillen, K.J. Painter, A user's guide to PDE models for chemotaxis, J. Math. Biol. 58 (2009) 183–217.
[25] B. Perthame, Transport Equations in Biology, in: Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2007.
[26] E.F. Keller, L.A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol. 26 (3) (1970) 399–415.
[27] C.S. Patlak, Random walk with persistence and external bias, Bull. Math. Biophys. 15 (1953) 311.
[28] J.T. Bonner, The Cellular Slime Molds, Princeton University Press, 1967.
[29] J.A. Carrillo, M. Fornasier, G. Toscani, F. Vecil, Particle, kinetic, and hydrodynamic models of swarming, in: Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences, in: Model. Simul. Sci. Eng. Technol., Birkhäuser Boston, Inc., Boston, MA, 2010, pp. 297–336.
[30] P. Carmeliet, Mechanisms of angiogenesis and arteriogenesis, Nat. Med. 6 (2000) 389–395.
[31] T.J. Newman, R. Grima, Phys. Rev. E 70 (2004) 051916.
[32] M.A. Herrero, E. Medina, J.J.L. Velázquez, Finite-time aggregation into a single point in a reaction–diffusion system, Nonlinearity 10 (6) (1997) 1739–1754.
[33] G. Wolansky, On steady distributions of self-attracting clusters under friction and fluctuations, Arch. Ration. Mech. Anal. 119 (4) (1992) 355–391.
[34] P.-H. Chavanis, C. Sire, Exact analytical solution of the collapse of self-gravitating Brownian particles and bacterial populations at zero temperature, Phys. Rev. E 83 (3) (2011) 031131.
[35] C. Sire, P.-H. Chavanis, Critical dynamics of self-gravitating Langevin particles and bacterial populations, Phys. Rev. E 78 (3) (2008) 061111.
[36] M. Doi, Molecular dynamics and rheological properties of concentrated solutions of rodlike polymers in isotropic and liquid crystalline phases, J. Polym. Sci. Polym. Phys. Ed. 19 (1981) 229–243.
[37] R.G. Larson, The Structure and Rheology of Complex Fluids, Oxford University Press, London, 1999.
[38] P. Constantin, I.G. Kevrekidis, E.S. Titi, Asymptotic states of a Smoluchowski equation, Arch. Ration. Mech. Anal. 174 (3) (2004) 365–384.
[39] Peter Constantin, Ioannis Kevrekidis, Edriss S. Titi, Remarks on a Smoluchowski equation, Discrete Contin. Dyn. Syst. 11 (1) (2004) 101–112.
[40] M. Alber, N. Chen, T. Glimm, P.M. Lushnikov, Phys. Rev. E 73 (2006) 051901.
[41] M. Alber, N. Chen, P.M. Lushnikov, S.A. Newman, Phys. Rev. Lett. 99 (2007) 168102.
[42] P.M. Lushnikov, N. Chen, M. Alber, Phys. Rev. E 78 (2008) 061904.
[43] T. Nagai, Blow-up of radially symmetric solutions to a chemotaxis system, Adv. Math. Sci. Appl. 5 (2) (1995) 581–601.
[44] P. Biler, Local and global solvability of some parabolic systems modelling chemotaxis, Adv. Math. Sci. Appl. 8 (2) (1998) 715–743.
[45] P.M. Lushnikov, Critical chemotactic collapse, Phys. Lett. A 374 (2010) 1678–1685.
[46] S.A. Dyachenko, P.M. Lushnikov, N. Vladimirova, Logarithmic-type scaling of the collapse of Keller–Segel equation, AIP Conf. Proc. 1389 (2011) 709–712.
[47] J. Ostriker, Astrophys. J. 140 (1964) 1056.
[48] S.I. Dejak, P.M. Lushnikov, I.M. Sigal, Blowup of the Keller–Segel system (in preparation).
[49] W. Beckner, Sharp Sobolev inequalities on the sphere and the Moser–Trudinger inequality, Ann. of Math. 138 (2) (1993) 213–242.
[50] E. Carlen, M. Loss, Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on Sn, Geom. Funct. Anal. 2 (1992) 90–104.
[51] M. Struwe, Equivariant wave maps in two space dimensions, Comm. Pure Appl. Math. 56 (2003) 815–823.
[52] W. Jäger, S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis, Trans. Amer. Math. Soc. 329 (2) (1992) 819–824.
[53] S. Gustafson, I.M. Sigal, Mathematical Concepts of Quantum Mechanics, third ed., Springer, 2011.
[54] C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers, Springer, 1999.
[55] Victor Barcilon, Eigenvalues of the one-dimensional Smoluchowski equation, J. Stat. Phys. 82 (1–2) (1996) 267–296.
[56] John P. Boyd, Andrei Natarov, A Sturm–Liouville eigenproblem of the fourth kind: a critical latitude with equatorial trapping, Stud. Appl. Math. 101 (4) (1998) 433–455.
[57] Susan N. Brown, An asymptotic expansion for the eigenvalues arising in perturbations about the Blasius solution, Appl. Sci. Res. 19 (1968–1969) 111–119.
[58] Arjen Doelman, Robert A. Gardner, Tasso J. Kaper, Stability analysis of singular patterns in the 1D Gray–Scott model: a matched asymptotics approach, Physica D 122 (1–4) (1998) 1–36.
[59] P.M. Lushnikov, Two mechanisms of surface wave generation: Kelvin–Helmholtz and Miles instabilities, Izv. Atmos. Ocean. Phys. 34 (1998) 370–377.
[60] Milton Abramowitz, Irene A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, ninth dover printing, tenth GPO printing ed., Dover, New York, 1964.
[61] Harry Bateman, Arthur Erdelyi, Higher Transcendental Functions Volume 1, McGraw Hill Book Company, New York, 1953.