Dynamic criterion for collapse

P. M. Lushnikov[1)]
L. D. Landau Institute of Theoretical Physics, Russian Academy of Sciences, 117334 Moscow, Russia

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A sufficient condition for wave collapse in the nonlinear Schrödinger equation is derived on the basis of an estimate of the kinetic energy. The collapse criterion can therefore be called dynamic. It is shown that the dynamic criterion is related to the strong collapse regime of the nonlinear Schrödinger equation. The analogy of the motion of a Newtonian particle in a potential, which is used in the derivation of the collapse criterion, makes it possible to extend the sufficient conditions for collapse to a number of other nonlinear partial differential equations. © *1995 American Institute of Physics.*

In many physical models of the interaction of nonlinear waves the collapses of waves or the formation of singularities in the solutions of nonlinear partial differential equations over a finite time play an important role as effective mechanisms of energy concentration. One of the most versatile models of this type is the nonlinear Schrödinger equation (NLE)

$$i\psi_t+\Delta\psi+|\psi|^{2\sigma}\psi=0, \quad \sigma>0, \tag{1}$$

which describes the self-action of wave packets in media with a nonlinearity of degree $2\sigma+1$. In the most common case $\sigma=1$ the NSE serves as a model of the propagation of a powerful laser pulse in a medium with a symmetry center,[1] the evolution of Langmuir waves in a plasma,[2] and so on. The NSE can be written in the Hamiltonian form

$$i\psi_t=\frac{\delta H}{\delta\psi^*}, \quad H=\int\left(|\nabla\psi|^2-\frac{1}{\sigma+1}|\psi|^{2\sigma+2}\right)d^Dr, \tag{2}$$

where the Hamiltonian H and the number of particles $N=\int|\psi|^2d^Dr$ are integrals of the motion, and D is the dimension of the space.

In quantum mechanics Eq. (1) is the Schrödinger equation for the wave function of a condensate of a weakly nonideal Bose gas with an attractive potential[3] $U=-|\psi|^{2\sigma}$. On this basis, we shall consider for the NSE the kinetic energy $\tilde{K}=X/N$ and the potential energy $\tilde{U}=-(\sigma+1)Y/N$, where

$$X=\int|\nabla\psi|^2d^Dr, \quad Y=\frac{1}{\sigma+1}\int|\psi|^{2\sigma+2}d^Dr.$$

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The existence of collapse in the NSE can be proved by analyzing the temporal evolution of the quantity $A=\int r^2|\psi|^2 d^D r$, which can be interpreted as the average width of the initial distribution $|\psi|$. Using Eq. (1), we obtain the following expression for the first time derivative

$$A_t = 2i \int x_\alpha \left(\frac{\partial \psi^*}{\partial x_\alpha} \psi - \frac{\partial \psi}{\partial x_\alpha} \psi^* \right) d^D r, \tag{3}$$

where summation over repeated indices is implied. Repeated differentiation of the quantity A_t with respect to the time and integration by parts yield the following relation — often called the virial theorem:

$$A_{tt} = 8 \int |\nabla \psi|^2 d^D r - \frac{4\sigma D}{\sigma+1} \int |\psi|^{2\sigma+2} d^D r = 4\sigma D H - 4(\sigma D - 2) \int |\nabla \psi|^2 d^D r. \tag{4}$$

In the case $\sigma D \geqslant 2$ it follows from the last relation that

$$A_{tt} \leqslant 4DH, \tag{5}$$

and for $H<0$ the positive-definite quantity A becomes negative over a finite time by virtue of Eq. (1). This means that a singularity appears in the solution of the given NSE.[2,4] In addition, the decrease in A near the collapse point indicates that the initial wave beam is self-focused (compressed). In the case $H>0$, $\sigma D \geqslant 2$ both collapse and spreading of the initial disturbance are possible. Although the NSE is no longer applicable near the formation point of a singularity and dissipative or some other limiting mechanisms come into play, the strong compression of the wave packet can be described on the basis of the NSE. Collapse is thus an effective mechanism of energy dissipation. In this connection, it is very important to be able to predict the presence or absence of collapse for different classes of initial conditions. Collapse is impossible for $\sigma D<2$ (Ref. 5). The value $\sigma D=2$ is critical and the function $A(t)$ is determined completely by the virial theorem (4). In what follows we shall study everywhere supercritical NSE with $D=3$, $\sigma=1$, which is most important for applications. All results, however, can be trivially extended to any supercritical ($\sigma D>2$) NSE. In the case at hand the sufficient condition for collapse $H<0$ was clarified considerably in Ref. 5. It was found that for $X>X_N$ the inequality

$$A_{tt} \leqslant 12(H - H_N) \tag{6}$$

follows from (4). Here X_N and H_N are the values of X and H on the main soliton solution of Eq. (1)

$$\psi_0 = \lambda R(\lambda \mathbf{x}) e^{i\lambda^2 t}, \quad -\lambda^2 R + \Delta R + R^3 = 0. \tag{7}$$

Here $\lambda^2 = N_0^2/N$, $N_0 = 18.94$, $H_N = N_0^2/N$, and $X_N = 3N_0^2/N$. At the same time it was shown that for $H<H_N$ and $X<X_N$ defocusing (i.e., spreading of the initial wave packet) occurs. The numerical calculations show that for $H>H_N$ both collapse and defocusing are possible. According to the condition (5), collapse will obviously occur if the following condition is satisfied:[5]

$$A_t|_{t=0} \leqslant -4\sqrt{HA|_{t=0}}, \quad H>0. \tag{8}$$

In this study we found a sufficient condition for collapse. The condition is based on the estimate of the kinetic energy in the NSE. With some degree of arbitrariness, the criterion for collapse which we found can thus be called a "dynamic" criterion, in contrast to the "static" criterion in Ref. 5, where the potential energy was estimated. The dynamic collapse criterion supplements in a natural manner the results of Ref. 5. Specifically, we will show on the basis of an investigation of the Gaussian initial conditions that for $H>H_N$ this criterion can also predict collapse when the inequality (8) does not hold. In addition, we will derive a relation between the dynamic criterion for collapse and the regime of the so-called strong collapse of the NSE.[6]

Setting $\psi = Re^{i\phi}$ $(R=|\psi|)$ and using the Cauchy–Bunyakovskiĭ inequality, we obtain the following expression from the virial theorem (3):

$$|A_t| = 4\left|\int x_\alpha \frac{\partial \phi}{\partial x_\alpha} R^2 d^D r\right| \leqslant 4\sqrt{\int r^2|\psi|^2 d^D r}\sqrt{\int (\nabla\phi)^2 R^2 d^D r}. \tag{9}$$

At the same time, we can rewrite X as follows:

$$X = \int |\nabla\psi|^2 d^D r = \int (\nabla R)^2 d^D r + \int (\nabla\phi)^2 R^2 d^D r, \tag{10}$$

and the number of particles N can be related to A as

$$N = \int |\psi|^2 d^D r = -\frac{2}{D}\int (\mathbf{x}, \nabla R) R \leqslant \frac{2}{D}\sqrt{\int r^2|\psi|^2 d^D r}\sqrt{\int (\nabla R)^2 d^D r}. \tag{11}$$

Using Eqs. (9)–(11) and recalling that we have set $D=3$ and $\sigma=1$, we obtain from Eq. (4) the main differential inequality

$$A_{tt} = 12H - 4\int |\nabla\psi|^2 d^3 r \leqslant 12H - 9\frac{N^2}{A} - \frac{A_t^2}{4A}. \tag{12}$$

The substitution $A = B^{4/5}$ gives the inequality

$$B_{tt} \leqslant 15HB^{1/5} - \frac{45}{4}\frac{N^2}{B^{3/5}}, \tag{13}$$

which has a simple mechanical analogy. To understand it, we write this inequality in the form

$$B_{tt} = 15HB^{1/5} - \frac{45}{4}\frac{N^2}{B^{3/5}} - g^2(t), \tag{14}$$

where $g^2(t)$ is an unknown nonnegative function of time. Then B is the coordinate of a "particle" whose motion is attributable to two forces: a conservative force $F_1 = -\partial U/\partial B$ with the potential

$$U(B) = -\tfrac{25}{2}HB^{6/5} + \tfrac{225}{8}N^2 B^{2/5} \tag{15}$$

and an additional force $f_2 = -g^2(t)$ which pulls the particle toward the origin $B=0$. Because of the influence of this force, the energy of the "particle" is time-dependent: $E(t) = (B_t^2/2) + U(B)$. Collapse occurs when to the "particle" reaches the origin in a finite period of time. If the "particle" reaches zero without the force $-g^2(t)$, then sooner or later it will reach zero if this additional negative force is taken into account. For $H<0$ the particle will always fall into the origin. In what follows we shall study everywhere the case $H>0$. The potential energy $U(B)$ will then contain a barrier with a maximum at the point $B=B_m$, where $B_m = (\frac{3}{4}N^2/H)^{5/4}$. The energy $E_m = U(B_m)$ is critical, since the particle can overcome the barrier only if $E(0)>E_m$. This gives the sufficient conditions for collapse:

a) for $B(t=0)<B_m$, $E(0)<E_m$ the "particle" cannot overcome the barrier on the right side, and it therefore always falls to zero over a finite period of time;

b) for $E(0)>E_m$, $B_t(t=0)<0$ the "particle" is guaranteed to overcome the barrier from right to left in the case $B(t=0)>B_m$; correspondingly, it will fall into zero over a finite time, and especially in the case $B(t=0)\leqslant B_m$.

We now consider the behavior of the solution of Eq. (14) in the limit $B\to 0$, $B_t<0$, corresponding to a collapsing solution. In this case the quantities $B^{6/5}$ and $B^{2/5}$ are vanishingly small. Ignoring these quantities, we obtain the following expression after integrating this equation twice:

$$B(t) = -\int_{t_0}^{t}\sqrt{2E(t_0) - \int_{t_0}^{t''} g^2(t')B_{t'}dt'dt''} \leqslant -\sqrt{2E(t_0)}(t-t_0),$$

where $B(t_0)=0$, $t\leqslant t_0$. Considering the variable A again, we find

$$A = B^{4/5} \leqslant (2E(t_0))^{2/5}(t_0-t)^{4/5}. \tag{16}$$

We note that, in contrast to the previously employed collapse criteria,[2,5] where the "particle" fell into zero with a finite rate A_t, here A_t goes to $-\infty$ at the point $t=t_0$. Thus, according to Eq. (6), as $A\to 0$, $A_t<0$ we have

$$A(t) \leqslant 12(H-H_N)t_0(t-t_0). \tag{17}$$

For this reason, the criterion (12) also gives a higher rate of fall of the "particle" into the origin of the coordinates than does (6). However, the time of falling into the origin and not the rate of fall itself is of greatest interest: $A(t_0)=0$. This time is generally different for Eqs. (17) and (16). We designate this time as t_{0a} and t_{0b} for Eqs. (16) and (17), respectively, and designate by $A^{(a)}$ and $A^{(b)}$ the corresponding solutions of Eqs. (6) and (12), where the inequality signs are replaced by equal signs. Let the initial conditions at $t=t_i$ be the same for both solutions:

$$A^{(a)}(t_i) = A^{(b)}(t_i), \quad A_t^{(a)}(t_i) = A_t^{(b)}(t_i) < 0. \tag{18}$$

Here $A^{(a,b)}(t_i)$ are assumed to be small enough so that for $t\leqslant t_i$ the dependences (17) and (16) are valid. Substituting them into expressions (18), we obtain $t_{0b}<t_{0a}$. For the same initial conditions the zero point is therefore reached more rapidly with the criterion (12) than with the criterion (6). We underscore the fact that this result was obtained only for the asymptotic region $A(t)\to 0$.

The regime (16) of the vanishing of A corresponds to the so-called strong collapse of Eq. (1) — quasiclassical compression of the wave packet.[6] In the strong-collapse regime a finite energy is trapped at the singularity, in contrast with weak collapse,[6] for which, formally speaking, no energy is trapped at the singular point. This stems from the fact that weak collapse is described by the self-similar solution

$$\psi = \frac{1}{(t_0-t)^{\frac{1}{2}+i\alpha}}\chi\left(\frac{r}{(t_0-t)^{\frac{1}{2}}}\right), \quad \alpha=0.545\ldots, \tag{19}$$

which is valid only in a bounded neighborhood of the singularity, and therefore a finite amount of energy, which is determined by the characteristic values of $|\psi|$ for which Eq. (1) is inapplicable, is trapped in the collapse.

The numerical and analytic studies showed that, in practice, weak collapse occurs and strong collapse is unstable.[6,7] However, the collapse criterion (12) obtained above "knows nothing" about the stability or instability of specific solutions of the NSE and indicates the existence of solutions that vanish more rapidly than does solution (19). Despite the fact that this criterion is sufficient but not necessary, it gives a lower estimate of the function $A(t)$ that is identical to the answer corresponding to the strong-collapse regime. In the special case of quasiclassical initial conditions strong collapse can be observed in a numerical calculation, where there is not enough time for the instability to destroy the time dependence[8] $A\sim(t_0-t)^{4/5}$.

In Ref. 5 it was shown that the main soliton solution (7) plays a special role. For this solution inequality (6) becomes an exact equality and the small variations of (7) can lead to both quasicollapsing and defocusing (spreading) solutions. We now find the initial conditions for which the differential inequality (12) becomes an exact equality. It is thus sufficient to note that from the quantum-mechanical viewpoint inequalities (9)–(11) are Heisenberg uncertainty relations. These inequalities become exact for the class of functions

$$\psi_0 = pe^{-\alpha r^2 - i\kappa r^2}, \tag{20}$$

where p and κ are arbitrary numbers, and $\alpha>0$. For the initial conditions (20) the quantities $A|_{t=0}$, $A_t|_{t=0}$, H, and N can be calculated explicitly. As an example, for $\kappa=0$ and $p^2/\alpha=2^{5/2}$ we obtain $H/H_N=16\pi^3/N_0^2\simeq 1.38\ldots$, $B|_{t=0}=B_m$, $B_t|_{t=0}=0$, i.e., the "particle" is located at the top of the barrier and has zero initial velocity. Holding the ratio p^2/α constant and setting $\kappa=\delta^2$, where δ^2 is an arbitrarily small positive number, we find that the initial velocity is $B_t|_{t=0}<0$ and that the criterion (12) guarantees the existence of collapse. Here the inequality (8) does not hold for the indicated values of the parameters p,κ, and α, since the initial "velocity" $B_t|_{t=0}$ can be assumed to be arbitrarily close to zero. For this reason, the criterion of Ref. 5 does not predict collapse in this case and this special form of the initial conditions shows the nontrivial nature of the dynamical criterion for collapse. At the same time, for the functions (20) there exists a region of parameters where $H/H_N\gg 1$, $\kappa>0$, and both the criterion (12) and the criterion of Ref. 5 predict collapse. Similar results can also be obtained for hyper-Gaussian initial conditions $\psi_0=p\exp(-\alpha r^n-i\kappa r^n)$, where $n>1$. Even in this case, however, the region of values of the parameters p,κ, and α, where only the dynamical criterion predicts collapse, changes rapidly as n increases. This region therefore depends strongly on the initial

conditions and, in contrast with the collapse criterion of Ref. 5, it apparently cannot generally be determined in terms of the integrals of motion H and N. Therefore, for $H>H_N$ a constructive method for determining the possibility of collapse is to determine whether inequality (8) is satisfied and to analyze the dynamical criterion (12). An exhaustive answer is given in Ref. 5 in the case $H\leqslant H_N$ and the dynamical criterion is superfluous, although even in this region it can predict the existence of collapse.

Sufficient conditions for collapse have been obtained for other equations besides the NSE: modifications of the Boussinesq equation,[9,10] the three-dimensional cubic Kadomtsev–Petviashvilli equation,[11] and the nonlinear Klein–Gordon equation.[12] By analogy with Eq. (12), all known majorizing differential inequalities employed in the derivation of the sufficient conditions for collapse can be reduced to Newton's equation for the motion of a particle with the coordinate R under the action of a potential force and an additional force $\alpha g(t)^2$ with a constant sign, where R is a positive-definite quantity, and $\alpha=\text{const}$. Two situations are possible, depending on the sign of α:

i) $\alpha<0$. If the positive-definite quantity R reaches zero over a finite time without the force $\alpha g^2(t)$, then the particle will certainly reach zero when this negative force is taken into account. The collapse criteria for the NSE from Ref. 5 and Eq. (12) correspond to this case.

ii) $\alpha>0$. If the particle coordinate R becomes infinite over a finite time without the positive force $\alpha g^2(t)$, then it will certainly become infinite when this force is added. The collapse criteria for the Boussinesq equation and the nonlinear Klein–Gordon equation correspond to this case.

We note that the conditions i) and ii) obtained above are more general than the previously considered conditions in Ref. 5, since they do not impose any restrictions on the sign of $R_t|_{t=0}$. As an example, this makes it possible to reject the condition $R_t|_{t=0}>0$ in Ref. 13. The condition ii) also generalizes the result of Ref. 14, according to which for $\alpha>0$ a sufficient condition for collapse is that the potential force acting on a particle must always be positive (of course, here it is assumed that the potential drops off sufficiently rapidly so as to produce collapse).

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[1)]e-mail: lushniko@landau.ac.ru

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