Test 1 Solutions

1. (30 points) Consider the homogeneous heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L,$$

subject to boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \ \frac{\partial u}{\partial x}(L,t) = 0,$$

and the initial condition

$$u(x,0) = f(x) = \begin{cases} 2 & \text{if } 0 < x < L/2, \\ 0 & \text{if } L/2 \le x < L \end{cases}$$

a)(5 points) Calculate equilibrium temperature distribution without solution of differential equation.
b) (20 points) Solve the time dependent problem and compare with part (a). For a full credit you have to perform separation of variables, obtain eigenvalues and eigenfunctions, analyze for which sign of the eigenvalue λ you have solution, derive the formula for the general solution by superposition and satisfy the initial condition. You don't need to show the orthogonality of eigenfunctions.

(a) this BC corresponds to conservation of thermal energy (insulated ends).

$\Rightarrow$ Equilibrium solution $u = c_1 = \text{const}$

$\Rightarrow \quad e = c\rho \int_0^L f(x)\,dx = c\rho \int_0^{L/2} 2 \cdot dx = c\rho L = c\rho \int_0^L c_1\,dx = c\rho c_1 L$

$\Rightarrow \quad u(x,\infty) = c_1 = 1$

(b) $u(x,t) = \phi(x)\,h(t)$

$$\phi \frac{dh}{dt} = k h \frac{d^2\phi}{dx^2} \Rightarrow \frac{\frac{dh}{dt}}{kh} = \frac{\frac{d^2\phi}{dx^2}}{\phi} = -\lambda = \text{const}$$

$$\frac{dh}{dt} = -kh\lambda \Rightarrow h = c\,e^{-k\lambda t}$$

ODE BVP $\begin{cases} \frac{d^2\phi}{dx^2} = -\lambda\phi \\ \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0 \end{cases}$

$\underline{\lambda > 0}$: $\phi(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$\phi'(x) = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda}\, x) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda}\, x)$

$\phi'(0) = 0 = -c_2 \sqrt{\lambda} \Rightarrow c_2 = 0$

$\phi'(L) = 0 = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} L) = 0 \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, \ldots$

$c_1 \neq 0$ - to have nontrivial sol.

$\lambda \neq 0$ by assumption

Eigenfunctions $\cos\left(\frac{n\pi}{L} x\right)$

$\underline{\lambda = 0}$ $\Rightarrow$ $\phi = c_1 x + c_2$

$\phi'(x) = c_1 = 0 \Rightarrow c_1 = 0$

$\Rightarrow$ $\phi = \text{const}$ - eigenfunction for $\lambda = 0$

$\underline{\lambda < 0}$: $\phi = c_1 e^{\sqrt{-\lambda}\, x} + c_2 e^{-\sqrt{-\lambda}\, x}$

$\phi' = c_1 \sqrt{-\lambda}\, e^{\sqrt{-\lambda}\, x} - c_2 \sqrt{-\lambda}\, e^{-\sqrt{-\lambda}\, x}$

$\phi'(0) = 0 = \sqrt{-\lambda}\,(c_1 - c_2) \Rightarrow c_1 = c_2$

$\phi'(L) = c_1 \sqrt{-\lambda}\left(e^{\sqrt{-\lambda} L} + e^{-\sqrt{-\lambda} L}\right) \neq 0 \;\; \forall \lambda < 0$ $\Rightarrow$ no sol.

Super-position : $u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

IC: $u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$

$A_0 = \frac{1}{L} \int_0^L f(x)\,dx = \frac{1}{L} \cdot 2 \cdot \frac{L}{2} = 1$

$A_n = \frac{2}{L} \int_0^L f(x) \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\, 2 \int_0^{L/2} \cos\frac{n\pi x}{L}\,dx = \frac{4}{L}\,\frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)\Big|_0^{\frac{L}{2}} = \frac{4}{n\pi} \begin{cases} 1, & n = 1, 5, \ldots \\ -1, & n = 3, 7, \ldots \\ 0 & \text{otherwise} \end{cases}$

$= \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right)$

2. (35 points) Solve Laplace's equation inside a semicircle of a radius a ($0 < r < a$, $0 < \theta < \pi$) subject to the boundary conditions:

$$\frac{\partial u}{\partial \theta}(r,0) = 0,\ \frac{\partial u}{\partial \theta}(r,\pi) = 0 \text{ and } u(a,\theta) = f(\theta).$$

For a full credit explain all steps (see Problem 1(b) for details).

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$u = \Phi(\theta)\, G(r)$$

$$\Rightarrow -\frac{\frac{d^2\Phi}{d\theta^2}}{\Phi} = \frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda = \text{const.}$$

$\Rightarrow$ ODE BVP
$$\begin{cases} \dfrac{d^2\Phi}{d\theta^2} = -\lambda\Phi & 0 < \theta < \pi \\ \Phi'(0) = \Phi'(\pi) = 0 \end{cases}$$

$\underline{\lambda > 0}$
$$\Phi = c_1\cos(\sqrt{\lambda}\,\theta) + c_2\sin(\sqrt{\lambda}\,\theta)$$
$$\Phi' = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,\theta) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,\theta)$$
$$\Phi'(0) = 0 = c_2\sqrt{\lambda} = 0 \quad \Rightarrow c_2 = 0$$
$$\Phi'(\pi) = 0 = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi) \Rightarrow \sqrt{\lambda}\,\pi = n\pi,\ n = 1, 2, \ldots$$
$$\Rightarrow \boxed{\lambda = n^2} \quad \text{Eigenfunctions } \cos(n\theta)$$

$\underline{\lambda = 0}$: $\Phi(\theta) = c_1\theta + c_2 \Rightarrow \Phi' = c_1 = 0$

$\Rightarrow$ eigenfunction $\Phi = \text{const}$

$\underline{\lambda < 0}$
$$\Phi(\theta) = c_1 e^{\sqrt{-\lambda}\,\theta} + c_2 e^{-\sqrt{-\lambda}\,\theta}$$
$$\Phi'(\theta) = +(\sqrt{-\lambda})\left(c_1 e^{\sqrt{-\lambda}\,\theta} - c_2 e^{-\sqrt{-\lambda}\,\theta}\right)$$

$$\Phi'(0) = \sqrt{-\lambda}\,(C_1 - C_2) = 0 \Rightarrow C_1 = C_2$$

$$\Phi'(\pi) = \sqrt{-\lambda}\, C_1 \big(\underbrace{e^{\sqrt{-\lambda}\pi} - e^{-\sqrt{-\lambda}\pi}}_{\neq 0}\big) = 0 \Rightarrow C_1 \text{ - only trivial solution.}$$

$$r^2 \frac{d^2 G}{dr^2} + r\frac{dG}{dr} - n^2 G = 0$$

If $n \neq 0 \Rightarrow \quad G = C_1 r^{n} + C_2 r^{-n}$

But $C_2 = 0$ because $|G(0)| < \infty$.

$\underline{n = 0} \Rightarrow \quad G = \bar{C}_1 \ln r + \bar{C}_2$, $\bar{C}_1 = 0$ because $|G(0)| < \infty$

$\Rightarrow$ Superposition

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos(n\theta)}$$

BC at $\underline{r = a}$: $\quad u(a,\theta) = f(\theta) = \sum_{n=0}^{\infty} A_n a^n \cos(n\theta)$

$$\Rightarrow \boxed{\begin{aligned} A_0 &= \frac{1}{\pi}\int_0^{\pi} f(\theta)\, d\theta \\ a^n A_n &= \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos(n\theta)\, d\theta, \quad n = 1, 2, \ldots \end{aligned}}$$

3. (35 points) Solve the following nonhomogeneous problem (suppose that all functions and all their derivatives are smooth so you can assume that all term-by term differentiations are allowed):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + e^{-3t}\sin\left(\frac{2\pi}{L}x\right), \quad 0 \le x \le L, \quad \text{assume that } 3 \ne k(2\pi/L)^2,$$

subject to

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x).$$

For a full credit explain all steps (see Problem 1(b) for details) except that you do not need to analyze the case $\lambda < 0$.

Start from solution of homogeneous problem

$$u_t = k u_{xx}, \quad u(0,t) = u(L,t) = 0$$

$$u = \phi(x) h(t)$$

Similar to ①, $\frac{dh}{dt} = -k h \lambda$

ODE BVP $\begin{cases} \frac{d^2\phi}{dx^2} = -\lambda\phi \\ \phi(0) = \phi(L) = 0 \end{cases}$

$\lambda > 0$ $\quad \phi = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

$\phi(0) = 0 = c_1 \Rightarrow \phi(L) = c_2 \sin(\sqrt{\lambda}L) = 0$

$\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$

Eigenfunction $\sin\left(\frac{n\pi x}{L}\right)$

$\lambda = 0$ $\quad \phi = c_1 x + c_2$

$\phi(0) = c_2 = 0 \Rightarrow \phi(L) = c_1 L = 0 \Rightarrow c_1 = 0$ – only trivial sol.

$\lambda < 0$ $\quad \phi = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$

$\phi(0) = c_1 + c_2 = 0 \Rightarrow \phi(L) = c_1 \underbrace{\left(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}\right)}_{\ne 0} \Rightarrow c_1 = 0$ – only trivial sol.

using the method of eigenfunction expansion we look at the sol. in the form:

$$\boxed{u(x,t)=\sum_{n=1}^{\infty} B_n(t)\sin\left(\frac{n\pi}{L}x\right)}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{dB_n}{dt}\sin\left(\frac{n\pi x}{L}\right)=\sum_{n=1}^{\infty}B_n\left(-k\left(\frac{n\pi}{L}\right)^2\right)\sin\left(\frac{n\pi x}{L}\right)+e^{-3t}\sin\left(\frac{2\pi}{L}x\right)$$

$$\Rightarrow \underline{n\neq 2}\qquad \frac{dB_n}{dt}=-B_n k\left(\frac{n\pi}{L}\right)^2 \Rightarrow \boxed{B_n(t)=B_n(0)e^{-k\left(\frac{n\pi}{L}\right)^2 t}\quad n\neq 2}$$

$$\underline{n=2}\qquad \frac{dB_2}{dt}=-B_2 k\left(\frac{2\pi}{L}\right)^2+e^{-3t}$$

Look for sol. in the form: $B_2(t)=C(t)e^{-k\left(\frac{2\pi}{L}\right)^2 t}$

$$\Rightarrow \frac{dC}{dt}e^{-k\left(\frac{2\pi}{L}\right)^2 t}=e^{-3t}$$

$$C=\underbrace{C(0)}_{B_2(0)}+\int_0^t e^{-3t'+k\left(\frac{2\pi}{L}\right)^2 t'}dt'$$

$$=B_2(0)+\frac{e^{-3t+k\left(\frac{2\pi}{L}\right)^2 t}-1}{-3+k\left(\frac{2\pi}{L}\right)^2}\Rightarrow \boxed{B_2(t)=B_2(0)e^{-k\left(\frac{2\pi}{L}\right)^2 t}+\frac{e^{-3t}-e^{-k\left(\frac{2\pi}{L}\right)^2 t}}{-3+k\left(\frac{2\pi}{L}\right)^2}}$$

$$\text{IC}:\quad u(x,0)=f(x)=\sum_{n=1}^{\infty}B_n(0)\sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \boxed{B_n(0)=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

4. Optional for extra credit (35 points) Consider the homogeneous heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L,$$

subject to boundary conditions

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0,$$

and the initial condition

$$u(x,0) = f(x) = 1 \text{ for } 0 < x < L.$$

Find solution of that initial/boundary value problem. For a full credit explain all steps (see Problem 1(b) for details).

$u = \Phi(x)\, h(t)$

Similar to 1b :

$\frac{dh}{dt} = -\lambda k h \Rightarrow h = c e^{-\lambda k t}$

ODE BVP $\begin{cases} \frac{d^2\Phi}{dx^2} = -\lambda \Phi \\ \Phi(0) = 0 \\ \frac{\partial \Phi}{\partial x}(L) = 0 \end{cases}$

$\underline{\lambda > 0}$

$\Phi = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$

$\Rightarrow \Phi(0) = C_1 = 0$

$\frac{d\Phi}{dx} = C_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x)$

$\frac{d\Phi}{dx}(L) = C_2 \sqrt{\lambda} \cos(\sqrt{\lambda} L) = 0$

$\Rightarrow \sqrt{\lambda} L = (n - \frac{1}{2})\pi, \quad n = 1, 2, \ldots$

$\Rightarrow \lambda = \frac{(n-\frac{1}{2})^2 \pi^2}{L^2}$ - eigenvalues

Eigenfunction: $\sin\left(\frac{(n-\frac{1}{2})\pi}{L} x\right)$

$\underline{\lambda = 0}$: $\phi = c_1 x + c_2$

$\phi(0) = c_2 = 0$

$\frac{d\phi(L)}{dx} = c_1 = 0 \Rightarrow$ only trivial sol.

$\underline{\lambda < 0}$ $\phi = c_1 e^{\sqrt{-\lambda} x} + c_2 e^{-\sqrt{-\lambda} x}$

$\phi(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$\frac{d\phi}{dx} = \sqrt{-\lambda}\left(c_1 e^{\sqrt{-\lambda} x} - (-c_1) e^{-\sqrt{-\lambda} x}\right)$

$\Rightarrow \frac{d\phi(L)}{dx} = \sqrt{-\lambda}\, c_1 \left(e^{+\sqrt{-\lambda} L} + e^{-\sqrt{-\lambda} L}\right) = 0$

$\Rightarrow c_1 = 0$ – only trivial sol.

Superposition:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) e^{-\frac{(n-\frac{1}{2})^2 \pi^2}{L^2} k t}}$$

IC: $u(x,0) = 1 = \sum_{n=1}^{\infty} B_n \sin \frac{(n-\frac{1}{2})\pi x}{L}$

$$B_n = \frac{2}{L} \int_0^L 1 \cdot \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) dx = \frac{-2}{L} \frac{L}{\pi (n-\frac{1}{2})} \cos\left(\frac{(n-\frac{1}{2})\pi x}{L}\right)\Bigg|_0^L$$

$$= \frac{-2}{(n-\frac{1}{2})\pi}\left[\cos\left((n-\tfrac{1}{2})\pi\right) - 1\right]$$

$$= + \frac{2}{\pi (n-\frac{1}{2})}$$