math 313 HW 01 Sols

1.2.5 Conservation law of chemical concentration:

$$\frac{d}{dt}\int_x^{x+\Delta x} u(x',t)\,dx' = \Phi(x,t) - \Phi(x+\Delta x,t) + \int_x^{x+\Delta x} \alpha\, u(x',t)\,(\beta - u(x',t))\,dx'$$

$\Delta x \to 0$, $\Phi(x,t) \equiv$ flux.

$$\Rightarrow \frac{\partial}{\partial t} u(x,t)\,\Delta x = -\frac{\partial \Phi(x,t)}{\partial x}\Delta x + \alpha\, u(x,t)\,(\beta - u(x,t))\,\Delta x$$

But $\Phi = -\kappa \frac{\partial u}{\partial x}$

$$\Rightarrow \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + \alpha\, u(\beta - u)$$

1.3.2 Heat flux:

$$\Phi = -K_0(x)\frac{\partial u}{\partial x}$$

Flux is continuous at $x=0$ which means that $\Phi(x_{0-},t) = \Phi(x_{0+},t)$

$$\Rightarrow K_0(x_{0-})\frac{\partial u}{\partial x}(x_{0-},t) = K_0(x_{0+})\frac{\partial u}{\partial x}(x_{0+},t)$$

Also if $K_0(x_{0-}) = K_0(x_{0+})$,

i.e. $K_0(x)$ is continuous at $x = x_0$

$$\Rightarrow \frac{\partial u}{\partial x}(x_{0-}, t) = \frac{\partial u}{\partial x}(x_{0+}, t),$$

i.e. $\frac{\partial u}{\partial x}$ is continuous at $x = x_0$

1.4.1a $Q = 0$, $u(0) = 0$, $u(L) = T$

using eq (1.4.17)

$$u(x) = T_1 + \frac{T_2 - T_1}{L} x = \frac{T}{L} x$$

1.4.1c $Q = 0$ $\frac{\partial u}{\partial x}(0) = 0$, $u(L) = T$

General steady-state sol:

$$\frac{d^2 u}{dx^2} = 0 \Rightarrow u(x) = C_1 x + C_2$$

$$\frac{\partial u(0)}{\partial x} = C_1 = 0$$

$$u(L) = 0x + C_2 = T \Rightarrow \boxed{u(x) = T}$$

1.4.1 f

$$\frac{Q}{K_0} = x^2, \quad u(0) = T, \quad \frac{\partial u}{\partial x}(L) = 0$$

General steady-state sol.

$$\frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q = 0$$

Because thermal properties of rod are constant $\Rightarrow$ $K_0 = \text{const.}$

$$K_0 \frac{d^2 u}{dx^2} + x^2 K_0 = 0$$

$$\frac{d^2 u}{dx^2} + x^2 = 0$$

General sol of nonhomogeneous problem is obtained by twice integration over x:

$$\boxed{u = -\frac{x^4}{12} + C_1 + C_2 x}$$

To satisfy BC:

$$u(0) = C_1 = T$$

$$\frac{\partial u}{\partial x}(L) = -\frac{x^3}{3} + C_2 \Big|_{x=L} = -\frac{L^3}{3} + C_2 = 0$$

$$\Rightarrow C_2 = \frac{L^3}{3}$$

$$\Rightarrow \boxed{u = -\frac{x^4}{12} + \frac{L^3}{3}x + T}$$

1.4.3

$u(0)=0$

$u(2)=0$

$c\rho = 1$ $\quad$ $c\rho = 2$

$K_0 = 1$ $\quad$ $K_0 = 2$

$Q = 1$ $\quad$ $Q = 0$

perfect thermal contact.

According to 1.3.2. it implies

$$(1)\ \begin{cases} u(1_+,t) = u(1_-,t) \\ \underbrace{K_0(1_-)}_{=1}\frac{\partial u}{\partial x}(1_-,t) = \underbrace{K_0(1_+)}_{=2}\frac{\partial u}{\partial x}(1_+,t) \end{cases}$$

$$\Rightarrow \frac{\partial u}{\partial x}(1_-,t) = 2\frac{\partial u}{\partial x}(1_+,t)$$

In rod 1:

$$1\cdot\frac{d^2u}{dx^2} + 1 = 0 \Rightarrow u = -\frac{x^2}{2} + C_1 + C_2 x$$

$$u(0) = C_1 = 0$$

$$(2)\ \begin{cases} \frac{\partial u(1_-,t)}{\partial x} = -x + C_2\Big|_{x=1} = -1 + C_2 \\ u(1_-,t) = -\frac{1}{2} + C_2 \end{cases}$$

In rod 2:

$$2 \cdot \frac{d^2 u}{dx^2} = 0 \Rightarrow u = C_3 + C_4 x$$

$$u(2) = C_3 + 2C_4 = 0 \Rightarrow C_3 = -2C_4$$

$$(3) \begin{cases} \dfrac{\partial u(1_+,t)}{\partial x} = C_4 \\ u(1_+,t) = -2C_4 + C_4 = -C_4 \end{cases}$$

Eqns. (1)-(3) has 6 unknowns C_2, C_4, $u(1_+,t)$, $u(1_-,t)$, $\frac{\partial u}{\partial x}(1_+,t)$, $\frac{\partial u}{\partial x}(1_-,t)$ and total 6 conditions.

So Excluding $u(1_\pm,t)$ and $\frac{\partial u}{\partial x}(1_\pm,t)$ we obtain:

$$\begin{cases} -1 + C_2 = 2C_4 \\ -\dfrac{1}{2} + C_2 = -C_4 \end{cases} \Rightarrow \begin{array}{l} C_2 = \dfrac{2}{3} \\ C_4 = -\dfrac{1}{6}, \; C_3 = -2C_4 = \dfrac{1}{3} \end{array}$$

$$\Rightarrow \boxed{\begin{array}{ll} u = -\dfrac{x^2}{2} + \dfrac{2}{3}x, & 0 < x < 1 \\ u = \dfrac{1}{3} - \dfrac{x}{6}, & 1 < x < 2 \end{array}}$$

1.4.10

$$u_t = u_{xx} + 4 \quad \Rightarrow \quad \begin{matrix} K = 1 \\ c\rho = 1 \end{matrix}$$

$$u(x, 0) = f(x)$$

$$\frac{\partial u}{\partial x}(0, t) = 5$$

$$\frac{\partial u}{\partial x}(L, t) = 6$$

Find total thermal energy $E(t)$ in a rod.

$$E = \int_0^L u(x,t)\,dx$$

Using eq (1.2.7):

$$\frac{dE}{dt} = \int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L \Big(-\frac{\partial \phi}{\partial x} + \underbrace{Q}_{=4}\Big)\,dx$$

$$= \phi(0) - \phi(L) + 4L$$

But $\phi(x) = -\frac{\partial u}{\partial x}$

$$\Rightarrow \phi(0) = -5$$

$$\phi(L) = -6$$

$$\Rightarrow \frac{dE}{dt} = -5 + 6 + 5L = 4L + 1, \quad \text{also } E(0) = \int_0^L f(x)\,dx$$

$$\Rightarrow \boxed{E(t) = \int_0^L f(x)\,dx + (4L+1)t}$$

2.3.1a

$$u_t = \frac{\kappa}{r}\partial_r(r\partial_r u)$$

$$u = \phi(r)G(t)$$

$$\Rightarrow \phi\frac{dG}{dt} = \frac{\kappa G}{r}\partial_r(r\partial_r\phi)$$

$$\Rightarrow \frac{\frac{dG}{dt}}{\kappa G} = \frac{\cancel{\kappa}1}{r\phi}\partial_r(r\partial_r\phi) = -\lambda$$

$$\Rightarrow \boxed{\begin{cases} \frac{dG}{dt} = -\kappa\lambda G \\ \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda\phi \end{cases}}$$

2.3.1c

$$u_{xx} + u_{yy} = 0$$

$$u = \phi(x)h(y)$$

$$\Rightarrow h\phi_{xx} + \phi h_{yy} = 0$$

dividing by ϕh:

$$\frac{\phi_{xx}}{\phi} = -\frac{h_{yy}}{h} = -\lambda \Rightarrow \boxed{\begin{array}{l} \frac{d^2\phi}{dx^2} = -\lambda\phi \\ \frac{d^2h}{dy^2} = \lambda h \end{array}}$$

2.3.2.b

(*) $$\frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\varphi(0) = 0 \quad , \varphi(1) = 0$$

General sol of (*) :

$\lambda > 0$ $$\varphi = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

$$\varphi(0) = c_1 = 0$$

$$\varphi(1) = c_2 \sin(\sqrt{\lambda}) = 0$$

$$\Rightarrow \sqrt{\lambda} = \pi n, \quad \lambda = n^2\pi^2, \; n = 1, 2, \ldots$$

$\lambda = 0$ $$\varphi = c_1 + c_2 x$$

$$\varphi(0) = c_1 = 0$$
$$\varphi(1) = c_2 = 0 \quad \Rightarrow \text{trivial sol.}$$

$\lambda < 0$

$\underline{\lambda < 0}$

$$\phi = c_1 e^{\sqrt{-\lambda}\, x} + c_2 e^{-\sqrt{-\lambda}\, x}$$

$$\phi(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$\phi(1) = c_1 \underbrace{\left(e^{\sqrt{-\lambda}\, 1} - e^{-\sqrt{-\lambda}\, 1}\right)}_{>0} \Rightarrow c_1 = 0$$

$\Rightarrow \phi \equiv 0$ - trivial sol only.

I.e. only solutions
for $\lambda = n^2 \pi^2$, $n = 1, 2, \ldots$

2.3.2.d

$$\phi(0) = 0, \quad \frac{d\phi}{dx}(L) = 0$$

$\underline{\lambda > 0}$ $\quad \phi = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$\phi(0) = 0 \Rightarrow c_1$

$$\frac{d\varphi(L)}{dx} = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)\Big|_{x=L} = 0$$

$$\Rightarrow \sqrt{\lambda}L = n\pi - \frac{\pi}{2}, \quad n = 1, 2, \ldots$$

$$\lambda = \frac{\left(n\pi - \frac{\pi}{2}\right)^2}{L^2}, \quad n = 1, 2 \ldots$$

$\underline{\lambda = 0}$ $\Rightarrow \phi = c_1 + c_2 x$

$$\phi(0) = c_1 = 0$$

$$\frac{d\phi(L)}{dL} = c_2 = 0 \Rightarrow \text{trivial sol.}$$

$\lambda = 0$ is <u>Not</u> eigenvalue

$\underline{\lambda < 0}$

$$\phi = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

$$\phi(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$\frac{d\phi(L)}{dL} = \sqrt{-\lambda}\left(c_1 e^{\sqrt{-\lambda}L} + c_1 e^{-\sqrt{-\lambda}L}\right)$$

$$= c_1\sqrt{-\lambda}\,\big(\underbrace{e^{\sqrt{-\lambda}L} + e^{-\sqrt{-\lambda}L}}_{\neq 0}\big) = 0 \Rightarrow c_1 = 0$$

$\Rightarrow$ only trivial solution $c_1 = c_2 = 0$

$\Rightarrow$ $\lambda < 0$ – Not the eigenvalue