

Hw 02 Solution 1

math 3/2 (1)

2.3.3.b

$$u_t = k u_{xx}$$

$$u(0, t) = u(L, t) = 0$$

$$u(x, 0) = 3 \sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)$$

From (2.3.30) of textbook:

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\text{IC: } u(x, 0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = 3 \sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)$$

$$\Rightarrow \text{by visual inspection } B_1 = 3, B_3 = -1, \\ B_2 = B_4 = B_5 = \dots = 0$$

or by (2.3.32):

$$B_n = \frac{2}{L} \int_0^L \left(3 \sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right) \right) \sin\left(\frac{n\pi x}{L}\right) dx$$

~~for $n=1$ we get~~

$$= \frac{2}{L} \int_0^L \frac{3}{L} \left[\cos\left(\frac{(1-n)\pi x}{L}\right) - \cos\left(\frac{(1+n)\pi x}{L}\right) \right] dx$$

$$= \frac{2}{L} \cdot 3 \cdot \frac{L}{2} \delta_{n,1} + \frac{2(-1)}{L} \frac{L}{2} \delta_{n,3}$$

$$= 3\delta_{n,1} - 1\delta_{n,3}, \text{ i.e. } B_1=3, B_3=-1, B_2=B_4=\dots=0$$

$$u(x,t) = 3 \sin\left(\frac{3\pi x}{L}\right) e^{-k\left(\frac{3\pi}{L}\right)^2 t} - \sin\left(\frac{\pi x}{L}\right) e^{-k\left(\frac{\pi}{L}\right)^2 t}$$

2.3.3.c $u(x,0) = 2 \cos\left(\frac{3\pi x}{L}\right)$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n = \frac{2}{L} \int_0^L 2 \cos\left(\frac{3\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

= } by $\cos \theta \cdot \sin \varphi = \frac{1}{2} \left[\begin{matrix} \sin(\theta + \varphi) \\ -\sin(\theta - \varphi) \end{matrix} \right]$ from Wiki }

$$= \frac{2}{L} \frac{2}{2} \int_0^L \left(\sin\left(\frac{(3+n)\pi x}{L}\right) - \sin\left(\frac{(3-n)\pi x}{L}\right) \right) dx$$

$$= \frac{2}{L} \left[\frac{L}{\pi(3+n)} \cos\left(\frac{(3+n)\pi x}{L}\right) \cdot \left(\delta_{n+3,0} + 1 \right) + \frac{L}{\pi(3-n)} \cos\left(\frac{\pi x}{L}\right) \cdot \left(\delta_{3-n,0} + 1 \right) \right]_{x=0}^L$$

$$= \frac{2}{\pi} \left[\frac{(-1)^{3+n} - 1}{3+n} (-1) + \frac{1}{3-n} [(-1)^{3-n} - 1] (1 - \delta_{3-n,0}) \right]$$

$$= \sum_{n=1}^{\infty} \left[\frac{(-1)^n + 1}{3+n} \rightarrow \frac{(-1)^{n+1}}{n-3} (1 - \delta_{3-n,0}) \right]$$

0 if n=3

$$= \frac{4}{\pi} \begin{cases} 0, & n \text{ odd} \\ \frac{1}{3+n} - \frac{1}{n-3}, & n \text{ even} \end{cases}$$

2.3.4 a

$$u_t = \kappa \frac{\partial^2 u}{\partial x^2}$$

$$u(0, t) = u(L, t) = 0, \quad u(x, 0) = f(x)$$

total heat energy = $\int_0^L c p u A dx$

$$= c p A \sum_{n=1}^{\infty} B_n e^{-\kappa \left(\frac{n\pi}{L}\right)^2 t} \int_0^L \sin \frac{n\pi x}{L} dx$$

$$= c p A \sum_{n=1}^{\infty} B_n e^{-\kappa \left(\frac{n\pi}{L}\right)^2 t} \frac{-\cos(n\pi) + 1}{\frac{n\pi}{L}}$$

$$= c p A \sum_{n=1}^{\infty} B_n e^{-\kappa \left(\frac{n\pi}{L}\right)^2 t} \frac{1 - (-1)^n}{\frac{n\pi}{L}}$$

where $B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$

according to (2.3.35)

2.3.4. b



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Heat flux to the right $= -k_0 \frac{\partial u}{\partial x}$

Total heat flow to the right $= -k_0 \frac{\partial u}{\partial x} A$

At $x=0$: heat flow out $= k_0 \frac{\partial u}{\partial x} \Big|_{x=0} A$

$$= k_0 A \sum_{n=1}^{\infty} B_n e^{-k \left(\frac{n\pi}{L}\right)^2 t} \frac{n\pi}{L} \cos\left(\frac{n\pi x}{L}\right) \Big|_{x=0}$$

$$= k_0 A \sum_{n=1}^{\infty} B_n e^{-k \left(\frac{n\pi}{L}\right)^2 t} \frac{n\pi}{L}$$

At $x=L$ heat flow out $= -k_0 \frac{\partial u}{\partial x} \Big|_{x=L} A$

$$= -k_0 A \sum_{n=1}^{\infty} B_n e^{-k \left(\frac{n\pi}{L}\right)^2 t} \frac{n\pi}{L} \cos(n\pi) \\ \parallel \\ (-1)^n.$$

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2.3.4.c

From conservation of

thermal energy

$$\frac{d}{dt} \int_0^L u dx = k \frac{\partial u}{\partial x} \Big|_0^L = k \frac{\partial u}{\partial x}(L) - k \frac{\partial u}{\partial x}(0)$$

rate of change of thermal energy

$k = \frac{k_0}{c_p}$ fluxes through ends

2.3.5

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \frac{1}{2} \int_0^L \left(\cos \frac{(n-m)\pi x}{L} - \cos \frac{(n+m)\pi x}{L} \right) dx$$

$$= \frac{1}{2} \left[\delta_{n,m} L + (1 - \delta_{n,m}) \frac{\sin \frac{(n-m)\pi x}{L}}{\frac{(n-m)\pi}{L}} \Big|_0^L - \delta_{n,-m} L + (1 - \delta_{n,-m}) \frac{\sin \frac{(n+m)\pi x}{L}}{\frac{(n+m)\pi}{L}} \Big|_0^L \right]$$

where $n, m > 0$

$$= \frac{L}{2} \delta_{n,m}$$

2.3.8

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$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} - \alpha u \quad , \quad \alpha > 0, \kappa > 0$$

$$u(0, t) = u(L, t) = 0.$$

(a) Equilibrium sol:

$$\kappa \frac{d^2 u}{dx^2} = \alpha u$$

General ODE solution:

$$u(x) = a \cosh\left(\sqrt{\frac{\alpha}{\kappa}} x\right) + b \sinh\left(\sqrt{\frac{\alpha}{\kappa}} x\right)$$

$$u(0) = a = 0$$

$$u(L) = b \sinh\left(\sqrt{\frac{\alpha}{\kappa}} L\right) = 0$$

$$\Rightarrow b = 0$$

$\Rightarrow u(x) \equiv 0$ - only equilibrium.

(b) Separation of variables:

$$u = \Phi(x) \cdot h(t)$$

$$\Rightarrow \Phi \frac{dh}{dt} + \alpha \Phi h = \kappa h \frac{d^2 \Phi}{dx^2}$$

$$\frac{\frac{dh}{dt}}{\kappa h} + \frac{\alpha}{\kappa} = \frac{1}{\Phi} \frac{d^2 \Phi}{dx^2} = -\lambda$$

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ODE BVP:

$$\begin{cases} \frac{d^2 \Phi}{dx^2} = -\lambda \Phi \\ \Phi(0) = \Phi(L) = 0 \end{cases} \quad \begin{array}{l} \text{Eigenvalue:} \\ \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, n=1, 2, \dots \\ \text{Eigen functions:} \end{array}$$

$$\Phi = \sin\left(\frac{n\pi x}{L}\right)$$

For h :

$$\frac{dh}{dt} + \alpha h = -\kappa h = -\kappa \left(\frac{n\pi}{L}\right)^2 h$$

$$\Rightarrow h = c e^{-\alpha t} \cdot e^{-\kappa \left(\frac{n\pi}{L}\right)^2 t}$$

Superposition:

$$u(x,t) = e^{-\alpha t} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\kappa \left(\frac{n\pi}{L}\right)^2 t}$$

$$IC: u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

orthogonality $\Rightarrow b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

As $t \rightarrow \infty \Rightarrow u \rightarrow 0$ as in equilibrium in (a).

⑧

2.4.1 a)

$$u_t = \kappa u_{xx}, \quad 0 < x < L, \quad t > 0.$$

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0, \quad t > 0$$

In text (2.4.13) u is found as:

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\kappa \left(\frac{n\pi}{L}\right)^2 t}$$

with (2.4.23-24)

$$\begin{cases} A_0 = \frac{1}{L} \int_0^L f(x) dx \\ A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \end{cases}$$

$$\Rightarrow (a) \quad u(x, 0) = \begin{cases} 0, & x < \frac{L}{2} \\ 1, & x > \frac{L}{2} \end{cases}$$

$$A_0 = \frac{1}{L} \int_{\frac{L}{2}}^L dx = \frac{1}{2}$$

$$A_n = \frac{2}{L} \int_{\frac{L}{2}}^L \cos \frac{n\pi x}{L} dx = \frac{2}{L} \cdot \frac{L}{n\pi} \left. \sin\left(\frac{n\pi x}{L}\right) \right|_{x=\frac{L}{2}}^L$$

$$= \frac{2}{n\pi} \left(\underbrace{\sin(n\pi)}_0 - \sin n \frac{n\pi}{2} \right) = -\frac{2}{n\pi} \begin{cases} 0, & n=2, 4, 6, \dots \\ 1, & n=1, 5, 9, \dots \\ -1, & n=3, 7, 11, \dots \end{cases}$$

(9)

$$2.4.1(b) \quad u(x,0) = 6 + 4 \cos \frac{3\pi x}{L}$$

by inspection: $A_0 = 6$, $A_3 = 4$,
 other $A_n = 0$.

$$2.4.1(c) \quad u = -2 \sin\left(\frac{\pi x}{L}\right)$$

$$A_0 = \frac{1}{L} \int_0^L -2 \sin \frac{\pi x}{L} dx = \frac{2}{\pi} \cos \frac{\pi x}{L} \Big|_0^L = \frac{2}{\pi} (1 + \cos \pi)$$

$$= -\frac{4}{\pi}$$

$$A_n = -\frac{4}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$\left\{ \text{use } \sin \theta \cos \varphi = \frac{1}{2} [\sin(\theta + \varphi) + \sin(\theta - \varphi)] \right\}$$

$$= -\frac{2}{L} \int_0^L \left[\sin\left(\frac{(n+1)\pi x}{L}\right) + \sin\left(\frac{(n-1)\pi x}{L}\right) \right] dx$$

$$= \frac{2}{L} \frac{L}{(n+1)\pi} \cos\left(\frac{(n+1)\pi x}{L}\right) \Big|_0^L - \frac{2}{L} \frac{L}{(n-1)\pi} \cos\left(\frac{(n-1)\pi x}{L}\right) \Big|_0^L \cdot \delta_{n,1}$$

$$= \frac{2}{(n+1)\pi} [(-1)^{n+1} - 1] - \frac{2}{(n-1)\pi} [(-1)^{n-1} - 1] \delta_{n,1}$$

$$= \begin{cases} -\frac{4}{(n+1)\pi} - \frac{4}{(n-1)\pi} \delta_{n,1}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

2.4.2

$$u_t = \kappa u_{xx},$$

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad u(L, t) = 0$$

$$u(x, 0) = f(x)$$

$$u = \phi(x) h(t)$$

$$\phi \cdot \frac{dh}{dt} = \kappa h \frac{d^2 \phi}{dx^2}$$

$$\Rightarrow \frac{\frac{dh}{dt}}{h} = \kappa \frac{\frac{d^2 \phi}{dx^2}}{\phi} = -\lambda$$

$$\frac{dh}{dt} = -\lambda h \Rightarrow h = C e^{-\lambda t}$$

$$\frac{d^2 \phi}{dx^2} = -\frac{\lambda}{\kappa} \phi \Rightarrow \phi = A \cos\left(\sqrt{\frac{\lambda}{\kappa}} x\right) + B \sin\left(\sqrt{\frac{\lambda}{\kappa}} x\right)$$

$$\Phi'(0) = 0 \Rightarrow -A \sqrt{\lambda}_k \sin\left(\sqrt{\lambda}_k x\right) + B \sqrt{\lambda}_k \cos\left(\sqrt{\lambda}_k x\right) \Big|_{x=0} \quad (11)$$

$$= B \sqrt{\lambda}_k = 0 \Rightarrow B = 0$$

$$\Phi(L) = A \cos\left(\sqrt{\lambda}_k L\right) = 0$$

$$\Rightarrow \sqrt{\lambda}_k L = \left(n + \frac{1}{2}\right) \pi, \quad n = 0, 1, \dots$$

$$\Rightarrow \lambda = \frac{\left(n + \frac{1}{2}\right)^2 \pi^2}{L^2} \quad k \quad - \text{eigenvalues}$$

$$\text{Eigen functions: } \cos\left(\frac{\left(n + \frac{1}{2}\right) \pi}{L} x\right)$$

Superposition:

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos\left[\frac{\left(n + \frac{1}{2}\right) \pi x}{L}\right] e^{-\frac{\left(n + \frac{1}{2}\right)^2 \pi^2 k t}{L^2}}$$

$$u(x, 0) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{\left(n + \frac{1}{2}\right) \pi x}{L}\right) = f(x)$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{\left(n + \frac{1}{2}\right) \pi x}{L}\right) dx, \quad n = 0, 1, \dots$$

~~$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{\left(n + \frac{1}{2}\right) \pi x}{L}\right) dx$$~~

2.4.3

$$\phi_{xx} = -\lambda \phi$$

$$\phi(0) = \phi(2\pi), \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi)$$

Set $y = x - \pi$

$$\Rightarrow \begin{cases} \phi_{yy} = -\lambda \phi \\ \phi(-\pi) = \phi(\pi) \\ \frac{d\phi(-\pi)}{dy} = \frac{d\phi(\pi)}{dy} \end{cases}$$

$L = \pi$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 = n^2, \quad n = 0, 1, \dots$$

$$\begin{aligned} \cos\left(\frac{n\pi y}{L}\right) &= \cos(ny) \\ &= \cos(n(x-\pi)) = (-1)^n \cos(nx) \end{aligned}$$

Eigenfunctions

$$\begin{aligned} \sin\left(\frac{n\pi y}{L}\right) &= \sin(ny) \\ &= \sin(n(x-\pi)) \\ &= (-1)^n \sin nx \end{aligned}$$

$(-1)^n$ can be included into arbitrary constant \Rightarrow eigenfunctions $\sin(nx), \cos nx$

2.5.1a

$$\Delta u = 0$$

$$0 \leq x \leq L, \quad 0 \leq y \leq H$$

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(L, y) = 0,$$

$$u(x, 0) = 0, \quad u(x, H) = f(x)$$

$$u = h(x) \phi(y)$$

$$\Rightarrow \frac{d^2 h}{dx^2} = -\frac{1}{\phi} \frac{d^2 \phi}{dy^2} = -\lambda$$

$$\Rightarrow \begin{cases} \frac{d^2 h}{dx^2} = -\lambda h \\ h'(0) = 0 \\ h'(L) = 0 \end{cases} \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2, \dots$$

$$h = \cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \frac{d^2 \phi}{dy^2} = \left(\frac{n\pi}{L}\right)^2 \phi$$

$$\underline{n=0}: \quad \phi = C_1 + C_2 y, \quad \phi(0) = 0 \Rightarrow C_1 = 0$$

$$\underline{n \neq 0} : \Phi = c_1 \cosh\left(\frac{n\pi y}{L}\right) + c_2 \sinh\left(\frac{n\pi y}{L}\right)$$

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$$\Phi(0) = c_1 = 0$$

Superposition:

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$$

the nonhomogeneous BC:

$$f(x) = u(x, H) = A_0 H + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi H}{L}\right) \cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow A_0 H = \frac{1}{L} \int_0^L f(x) dx$$

$$A_n \sinh\left(\frac{n\pi H}{L}\right) = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

(2.5.1.e)

$$u(0, y) = u(L, y) = 0$$

$$u(x, 0) = \frac{\partial u}{\partial y}(x, 0)$$

$$u(x, H) = f(x)$$

$$u = \phi(x) h(y)$$

$$\frac{d^2 \phi(x)}{dx^2} = - \frac{d^2 h}{dy^2} = -\lambda$$

ODE BVP:

$$\begin{cases} \frac{d^2 \phi}{dx^2} = -\lambda \phi \\ \phi(0) = \phi(L) = 0 \end{cases}$$

Eigenvalues

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$$

Eigenfunctions:

$$\phi = \sin \frac{n\pi x}{L}, \quad n=1, 2, \dots$$

$$\begin{cases} \frac{d^2 h}{dy^2} = \left(\frac{n\pi}{L}\right)^2 h \\ h(0) = \frac{dh}{dy}(0) \end{cases}$$

general sol

$$h = c_1 \cosh\left(\frac{n\pi y}{L}\right) + c_2 \sinh\left(\frac{n\pi y}{L}\right)$$

$$h(0) - \frac{dh}{dy}(0) = c_1 - \frac{n\pi}{L} \left[\sinh\left(\frac{n\pi y}{L}\right) \right]$$

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$$+ c_2 \cosh\left(\frac{n\pi y}{L}\right) \Big|_{y=0}$$

$$= c_1 - \frac{n\pi}{L} c_2 = 0 \Rightarrow c_1 = c_2 \frac{n\pi}{L}$$

$$\Rightarrow h = \cosh\left(\frac{n\pi y}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi y}{L}\right)$$

Superposition

$$u(x, y) = \sum_{n=1}^{\infty} A_n \left[\cosh\left(\frac{n\pi y}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi y}{L}\right) \right] \times \sin\left(\frac{n\pi x}{L}\right), \text{ where } A_n \text{ is}$$

$n=1, 2, \dots$

determined from

$$BC \quad f(x) = \sum_{n=1}^{\infty} A_n \left[\cosh\left(\frac{n\pi H}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi H}{L}\right) \right] \times \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow A_n \left[\cosh\left(\frac{n\pi H}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi H}{L}\right) \right] = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$