math 312 Hw 03 Solutions

A1

Problem 1

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

From class:

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial\theta\partial r} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} \qquad (1)$$

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \sin^2\theta\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}$$

$$+ \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\,\partial\theta}$$

$$+ \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial\theta\,\partial r} - \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta}$$

$$+ \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} \qquad (2)$$

$$(1)+(2) = (\cos^2\theta + \sin^2\theta)\frac{\partial^2}{\partial r^2} + (\sin^2\theta + \cos^2\theta)\frac{1}{r}\frac{\partial}{\partial r}$$

$$+ \frac{(\cos^2\theta + \sin^2\theta)}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$\Rightarrow \boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}u\right) + \frac{1}{r}\frac{\partial^2 u}{\partial\theta^2} = 0}$$

math 312

HW 03 Solutions

2.5.2a

$0 < x < L$
$0 < y < H$

$$\frac{\partial u}{\partial x}(0,y)=0 \quad , \quad \frac{\partial u}{\partial y}(x,0)=0$$

$$\frac{\partial u}{\partial x}(L,y)=0 \quad , \quad \frac{\partial u}{\partial y}(x,H)=f(x)$$

(a) the total heat flow across the boundary must be equal to zero in equilibrim (i.e. without sources in Laplace eq).

$$\Rightarrow \int_0^L f(x)\,dx = 0$$

(b) using (2.5.16): $u = h(x)\,\Phi(y)$

$$\frac{h_{xx}}{h} = -\frac{\Phi_{yy}}{\Phi} = -\lambda$$

$$\begin{cases} h_{xx} = -\lambda h \\ \frac{\partial h}{\partial x}(0) = \frac{\partial h}{\partial x}(L) = 0 \end{cases}$$

homogeneous BVP

~~$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3, \ldots$~~

~~$h(x) = \cos\frac{n\pi}{L}x$ eigenfunctions~~

②

$$h = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

$$h_x = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda}\, x) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda}\, x)$$

$$h_x(0) = c_2 \sqrt{\lambda} = 0 \Rightarrow c_2 = 0$$

$$h_x(L) = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} L) = 0$$

$$\Rightarrow \lambda = 0 \text{ or } \sqrt{\lambda} L = n\pi, \quad n = 0, 1 \ldots$$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2 \ldots$$

Eigenfunctions: $\cos\left(\frac{n\pi}{L} x\right)$

$\lambda = 0$: $h = c_1 + c_2 x$

$h_x = c_2 = 0$

$$\begin{cases} \dfrac{d^2 \varphi}{dy^2} = \left(\dfrac{n\pi}{L}\right)^2 \varphi \\ \dfrac{d\varphi}{dy}(0) = 0 \end{cases}$$

$$\Rightarrow \varphi = \cosh\left(\frac{n\pi}{L} y\right)$$

Superposition:

$$u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right)$$

$$\frac{\partial u(x,H)}{\partial y} = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi H}{L}\right) = f(x)$$

$$\Rightarrow A_n \cosh\left(\frac{n\pi H}{L}\right) = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad n = 1, 2, \ldots$$

$$u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right),$$

(*) A_0 - arbitrary.

((c (optional)) : to find A_0

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u, \qquad u(x,y,0) = g(x,y)$$

Initial heat energy: $E(0) = c\rho \int_0^L \int_0^H u(x,y,0)\,dx\,dy$

$$= c\rho \int_0^L \int_0^H g(x,y)\,dx\,dy$$

Rate of change of heat energy: $\frac{dE}{dt} = -\int_0^L K_0 \frac{\partial u}{\partial y}(x,H)\,dx = -K_0 \int_0^L f(x)\,dx$

$= 0$ according to (a)

$$\Rightarrow E(t)\Big|_{t\to\infty} = E(0) = c\rho \int_0^L \int_0^H g(x,y)\,dx\,dy$$

But $E(\infty) = c\rho \int_0^L \int_0^H u(x,y)\,dx\,dy$ (using solution $(*)$ of (γ))

$$= c\rho \cdot L \cdot H \cdot A_0 + 0$$

$$\Rightarrow A_0 = \frac{\int_0^L \int_0^H u(x,y)\,dx\,dy}{LH}$$

2.5.3 Outside of a circular disk $r \geq a$

$$\nabla^2 u = \frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_\theta^2 u = 0$$

$$u = \phi(\theta)\,G(r)$$

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda$$

⑤

$$\begin{cases} \dfrac{d^2\Phi}{d\theta^2} = -\lambda \Phi \\ \Phi(-\pi) = \Phi(\pi) \\ \dfrac{d\Phi}{d\theta}(-\pi) = \dfrac{d\Phi}{d\theta}(\pi) \end{cases}, \quad l = \pi$$

$\Rightarrow \lambda = \left(\dfrac{n\pi}{l}\right)^2 = n^2$ – eigenvalues

$\sin(n\theta), \cos(n\theta)$ – eigenfunctions.

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = n^2$$

$$r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2 G = 0$$

$n \neq 0$: $G = c_1 r^n + c_2 r^{-n}$

$n = 0 \Rightarrow G = \bar{c}_1 + \bar{c}_2 \ln r$

to have $|u| < \infty$ as $r \to \infty$ $\Rightarrow c_1 = \bar{c}_2 = 0$

Superposition:

$$u = \sum_{n=0}^{\infty} A_n r^{-n}\cos(n\theta) + \sum_{n=1}^{\infty} B_n r^{-n}\sin(n\theta)$$

(a) $u(a,\theta) = \ln 2 + 4\cos(3\theta)$

$\Rightarrow A_0 = \ln 2$, $A_3 a^{-3} = 4$, $A_n = B_n = 0$ for other n

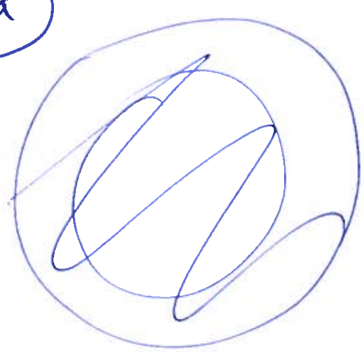
(b) $u(a,\theta) = f(\theta)$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$A_n a^{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$B_n a^{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta$$

2.5.6.a



$0 < \theta < \pi$

$0 < r < a$

$$\nabla^2 u = 0$$

$u = 0$ at diameter and $u(a,\theta) = g(\theta)$

Similar to (2.5.37) but different BC:

$$\begin{cases} \dfrac{d^2\Phi}{d\theta^2} = -\lambda\Phi \\ \Phi(0) = \Phi(\pi) = 0 \end{cases} \Rightarrow \lambda = \left(\frac{n\pi}{\pi}\right)^2 = n^2 \text{ eigenvalues}$$

$\sin(n\theta)$, $n = 1, 2, 3 \ldots$ – eigenfunctions.

$G = c_1 r^n + c_2 r^{-n}$, $c_2 = 0$ to have $|u| < \infty$ at $r=0$

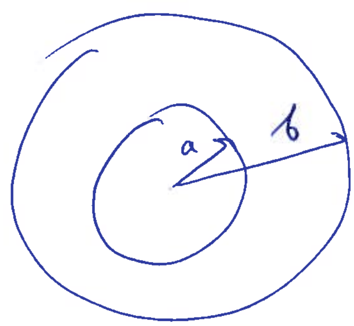
Superposition:

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin(n\theta)}$$

BC at $r=a$: $g(\theta) = \sum_{n=1}^{\infty} A_n a^n \sin(n\theta)$

$$\Rightarrow \boxed{A_n a^n = \frac{2}{\pi}\int_0^{\pi} g(\theta)\sin(n\theta)\,d\theta}$$

2.5.8 a



$a < r < b$

$\nabla^2 u = 0$

BC: $u(a,\theta) = f(\theta)$

$u(b,\theta) = g(\theta)$

$u = \Phi(\theta)\, G(r) \Rightarrow$

$$\nabla^2 u = \frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0$$

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\Phi}\frac{d^2\Phi}{d\theta^2} = \lambda$$

$-\pi < \theta < \pi$

Periodicity in φ:
$$\begin{cases} \varphi(\pi) = \varphi(-\pi) \\ \frac{d\varphi}{d\theta}(\pi) = \frac{d\varphi}{d\theta}(-\pi) \\ \frac{d^2\varphi}{d\theta^2} = -\lambda\varphi \end{cases}$$

$$\Rightarrow \varphi = c_1 \cos(\sqrt{\lambda}\,\theta) + c_2 \sin(\sqrt{\lambda}\,\theta)$$

$$\frac{d\varphi}{d\theta} = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,\theta) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,\theta)$$

B.C.: $\cancel{c_1\cos(\sqrt{\lambda}\,\pi)} + c_2\sin(\sqrt{\lambda}\,\pi) = \cancel{c_1\cos(\sqrt{\lambda}\,\pi)} - c_2\sin(\sqrt{\lambda}\,\pi)$

$$-c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi) + \cancel{c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi)} = c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi) + \cancel{c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi)}$$

$$\Rightarrow \sin(\sqrt{\lambda}\,\pi) = 0 \Rightarrow \sqrt{\lambda}\,\pi = n\pi, \quad n \in \mathbb{Z}$$

$\lambda = n^2$, $n = 0, 1, \ldots$ – eigenvalues

$\cos(n\theta)$, $\sin(n\theta)$ – eigenfunctions

~~$\lambda \neq 0$~~

~~$\lambda = 0 \Rightarrow \bar{c}_1 + \bar{c}_2 \ln x$~~

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = n^2 \Rightarrow r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2 G = 0$$

$\underline{\lambda \neq 0}$ $G = c_1 r^n + c_2 r^{-n}$, $n = 1, 2, \ldots$

$\underline{\lambda = 0}$ $G = \bar{c}_1 + \bar{c}_2 \ln r$

It is convenient to choose instead $r^{\pm n}$ the following eigenfunctions:

$$G_1(r) = \begin{cases} \ln \frac{r}{a}, & n = 0 \\ \left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n, & n \neq 0 \end{cases} \Rightarrow G_1(a) = 0$$

$$G_2(r) = \begin{cases} \ln \left(\frac{r}{b}\right), & n = 0 \\ \left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^n, & n \neq 0 \end{cases} \Rightarrow G_2(b) = 0$$

Superposition:

$$u(r, \theta) = \sum_{n=0}^{\infty} \cos(n\theta)\left[A_n \cancel{\Phi_1} G_1(r) + B_n G_2(r)\right]$$

$$+ \sum_{n=1}^{\infty} \sin(n\theta)\left[C_n G_1(r) + D_n \Phi_2(r)\right]$$

$\underline{BC}$: $r = a$: $f(\theta) = \sum_{n=0}^{\infty} \cos(n\theta)\left[B_n G_2(a)\right]$

$$+ \sum_{n=1}^{\infty} \sin(n\theta)\left[D_n G_2(a)\right]$$

$\underline{r = b}$ $g(\theta) = \sum_{n=0}^{\infty} \cos(n\theta)\left[A_n G_1(b)\right]$

$$+ \sum \sin(n\theta)\left[C_n G_1(b)\right]$$

$\Rightarrow$ By orthogonality

$$B_0 G_2(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta.$$

$$B_n G_2(a) = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(n\theta)\, f(\theta)\, d\theta \quad , n = 1, 2, \ldots$$

$$A_0 G_1(b) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, d\theta$$

$$A_n G_1(b) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta) \cos(n\theta)\, d\theta \quad , n = 1, 2, \ldots$$

$$C_0 G_1(b) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, d\theta$$

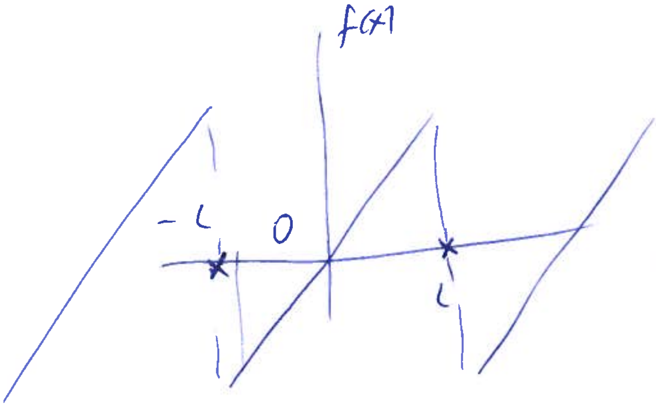
$$C_n G_1(b) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\theta)\, g(\theta)\, d\theta$$

$$D_0 G_2(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$D_n G_2(a) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\theta)\, f(\theta)\, d\theta$$

3.2.2.a

$$-L \le x \le L$$



$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\frac{x^2}{2}\Big|_{-L}^{L} = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx = 0$$ since x is odd.

$$b_n = \frac{2L}{\pi n}(-1)^{n+1} \quad \text{(from class)}$$

$$\Rightarrow \quad x \sim \sum_{n=1}^{\infty} \frac{2L}{\pi n}(-1)^{n+1}\sin\left(\frac{n\pi}{L}x\right).$$