

math 517

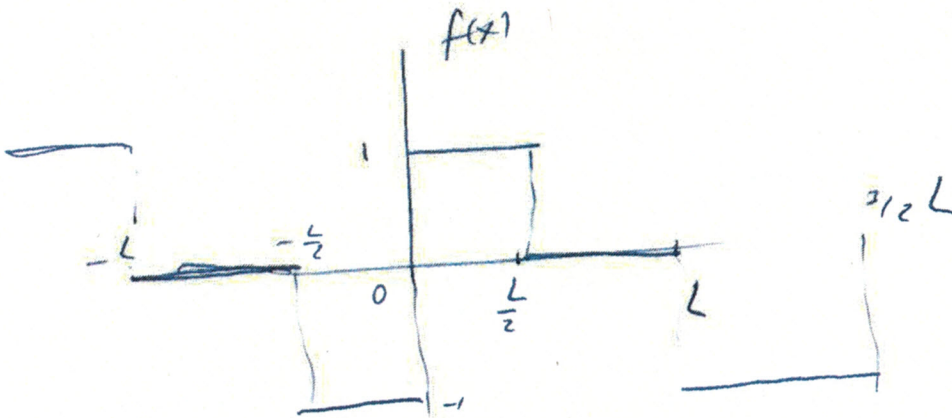
HW 04

3.3.2d

$$f(x) = \begin{cases} 1 & , x < \frac{L}{2} \\ 0 & , x > \frac{L}{2} \end{cases}$$

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Odd extension



$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \int_0^{L/2} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \left(\frac{L}{n\pi} \right) \cos\left(\frac{n\pi x}{L}\right) \Big|_0^{L/2}$$

$$= \frac{2}{n\pi} \left(1 - \cos\left(\frac{n\pi}{2}\right) \right)$$

3.3.10

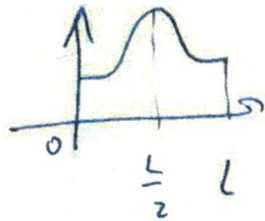
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$$f(x) = \begin{cases} x^2, & x < 0 \\ e^{-x}, & x > 0 \end{cases} \Rightarrow f(-x) = \begin{cases} x^2, & -x < 0 \\ e^x, & -x > 0 \end{cases}$$

$$f_e(x) = \frac{1}{2} [f(x) + f(-x)] = \frac{1}{2} \begin{cases} x^2 + e^x, & x < 0 \\ x^2 + e^{-x}, & x > 0 \end{cases}$$

$$f_o(x) = \frac{1}{2} [f(x) - f(-x)] = \frac{1}{2} \begin{cases} x^2 - e^x, & x < 0 \\ -x^2 + e^{-x}, & x > 0 \end{cases}$$

3.3.13



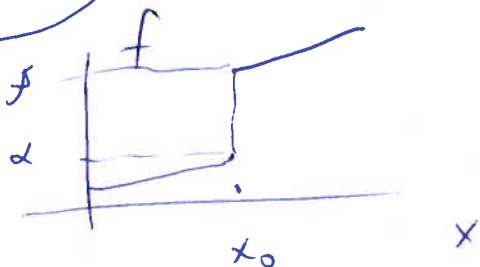
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

But $\sin\left(\frac{n\pi}{L}x\right)$ is odd around $\frac{L}{2}$ for n even

$\Rightarrow f(x) \sin\left(\frac{n\pi}{L}x\right)$ is odd around $\frac{L}{2}$ for

n even $\Rightarrow b_n = 0$ for n even.

3.4.3a



$$f \sim \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad a_n = \frac{2}{L} \int_0^L f \cos \frac{n\pi x}{L} dx \quad n \neq 0$$

We want to find:

$$f \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

Use integration by parts:

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L \frac{df}{dx} \sin \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} df \\ &= \frac{2}{L} \left[f \sin \frac{n\pi x}{L} \Big|_0^{x_0^-} + f \sin \frac{n\pi x}{L} \Big|_{x_0^+}^L \right. \\ &\quad \left. - \frac{n\pi}{L} \int_0^L f \cos \frac{n\pi x}{L} dx \right] = \frac{2}{L} \left[\alpha \sin \left(\frac{n\pi x_0}{L} \right) - 0 \right. \\ &\quad \left. + 0 - \beta \sin \left(\frac{n\pi x_0}{L} \right) + \frac{n\pi L a_n}{2} \right] \end{aligned}$$

$$= \sum \frac{2}{L} \sin\left(\frac{n\pi x_0}{L}\right) (\alpha - \beta) - \frac{n\pi}{L} a_n$$

(2)

3.4.9

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x, t)$$

$$u(0, t) = 0$$

$$u(L, t) = 0$$

$q(x, t)$ - piecewise smooth in $x \forall t$.

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L} \quad (*)$$

Because $\frac{\partial^2 u}{\partial x^2}$ is piecewise smooth and $u(0, t) = 0 = u(L, t)$

\Rightarrow and $u(x, t)$ is continuous

\Rightarrow we can differentiate (*) term by term:

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right) b_n \cos\left(\frac{n\pi x}{L}\right) \quad (**)$$

Because $\frac{\partial^2 u}{\partial x^2}$ is piecewise smooth

\Rightarrow we can differentiate (**) term by term:

(3)

$$\frac{\partial^2 u}{\partial x^2} = - \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} \right)^2 b_n \sin \frac{n\pi x}{L}$$

thus from p. 119 and from class:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{d b_n}{dt} \sin \frac{n\pi x}{L} = -k \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} \right)^2 b_n \sin \frac{n\pi x}{L} + q$$

$$\Rightarrow \frac{d b_n}{dt} + k \left(\frac{n\pi}{L} \right)^2 b_n = \frac{2}{L} \int_0^L q(x,t) \sin \left(\frac{n\pi x}{L} \right) dx$$

3.7.12

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + e^{-t} + e^{-2t} \cos \frac{3\pi x}{L}$$

(assume $2 \neq k \left(\frac{3\pi}{L} \right)^2$)

BC: $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$

IC: $u(x,0) = f(x)$

the eigenfunctions of homogeneous

O2E BVP ~~problem~~ are $\cos \frac{n\pi x}{L}$, $n=0, 1, 2, \dots$

$$\Rightarrow u \sim \sum_{n=0}^{\infty} A_n(t) \cos\left(\frac{n\pi x}{L}\right) - \text{it can}$$

be differentiated (if u is continuous)

since it is a cosine series:

$$\frac{\partial u}{\partial x} \sim \sum_{n=0}^{\infty} A_n\left(-\frac{n\pi}{L}\right) \sin\frac{n\pi x}{L}$$

this can be differentiated again

(if $\frac{\partial u}{\partial x}$ is continuous) only need

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0 - \text{according to BC}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} \sim - \sum_{n=0}^{\infty} A_n \left(\frac{n\pi}{L}\right)^2 \cos\left(\frac{n\pi x}{L}\right)$$

\Rightarrow { similar to problem 3.4.2 } :

$$\frac{\partial u}{\partial t} + k \frac{\partial^2 u}{\partial x^2} = \sum_{n=0}^{\infty} \left[\frac{dA_n}{dt} + k \left(\frac{n\pi}{L}\right)^2 A_n \right] \cos\frac{n\pi x}{L} = \underbrace{e^{-t} + e^{-2t} \cos\left(\frac{3\pi x}{L}\right)}_{\text{cos series in } x}$$

$$(\ast \ast \ast) \Rightarrow \frac{dA_n}{dt} + k \left(\frac{n\pi}{L}\right)^2 A_n = \begin{cases} e^{-t}, & n=0 \\ e^{-2t}, & n=3 \\ 0, & n \neq 0, n \neq 3 \end{cases}$$

(5)

IC gives $A_0(0) = \frac{1}{L} \int_0^L f(x) dx$

$A_n(0) = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, n \neq 0$

O.D.E. (***) is solved as follows:

$n=0$: $\frac{dA_0}{dt} = e^{-t} \Rightarrow A_0(t) - A_0(0) = \int_0^t e^{-t'} dt' = 1 - e^{-t}$

$$A_0(t) = A_0(0) + 1 - e^{-t}$$

$n \neq 0, n \neq 3$: $\frac{dA_n}{dt} = -\kappa \left(\frac{n\pi}{L}\right)^2 A_n$

$$\Rightarrow A_n = A_n(0) e^{-\kappa \left(\frac{n\pi}{L}\right)^2 t}$$

$n=3$ $A_3 = c(t) e^{-\kappa \left(\frac{3\pi}{L}\right)^2 t}$

$$\Rightarrow \frac{dc}{dt} e^{-\kappa \left(\frac{3\pi}{L}\right)^2 t} = e^{-2t}$$

$$c = \underbrace{c(0)}_{A_3(0)} + \int_0^t e^{-2t' + \kappa \left(\frac{3\pi}{L}\right)^2 t'} dt' = A_3(0)$$

$$+ \frac{e^{-2t + \kappa \left(\frac{3\pi}{L}\right)^2 t} - 1}{-2 + \kappa \left(\frac{3\pi}{L}\right)^2}$$

$$\Rightarrow A_3 = A_3(0) e^{-\kappa \left(\frac{3\pi}{L}\right)^2 t} + \frac{e^{-2t} - e^{-\kappa \left(\frac{3\pi}{L}\right)^2 t}}{-2 + \kappa \left(\frac{3\pi}{L}\right)^2}$$

3.5.1c

$$x^2 \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$\int_0^x dx \Rightarrow$$

$$\frac{x^3}{3} = \sum_{n=1}^{\infty} b_n \left(\frac{L}{n\pi} \right) \left(-\cos \left(\frac{n\pi x}{L} \right) \right) \Big|_0^x$$

$$= \sum_{n=1}^{\infty} b_n \left(-\frac{L}{n\pi} \right) \cos \left(\frac{n\pi x}{L} \right) + \sum_{n=1}^{\infty} b_n \left(\frac{L}{n\pi} \right)$$

Also according to (3.5.11)

~~$$\sum_{n=1}^{\infty} \frac{b_n}{L} \int_0^L \sin \frac{n\pi x}{L} dx = \frac{A}{3}$$~~

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3.5.7

Evaluate (3.5.6) at $x = \frac{L}{2}$:

$$\frac{x^2}{2} \Big|_{\frac{L}{2}} = \frac{L^2}{8} = \frac{L}{2} x - \frac{4L^2}{\pi^3} \sum_{n \text{ odd}} \sin\left(\frac{n\pi x}{L}\right) \Big|_{x=\frac{L}{2}}$$

$$= \frac{L^2}{4} - \frac{4L^2}{\pi^3} \left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \dots \right)$$

$$\Rightarrow \left[1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots \right] = -\frac{L^2}{8} \cdot \left(-\frac{\pi^3}{4L^2} \right)$$

$$= \frac{\pi^3}{32}$$

3.6.1

$$f(x) = \begin{cases} 0 & , x < x_0 \\ \frac{1}{\Delta} & , x_0 < x < x_0 + \Delta \\ 0 & , x > x_0 + \Delta \end{cases}$$

\Rightarrow using (3.6.7)

$$C_n = \frac{1}{2L} \int_{-L}^L f(x) e^{i \frac{n\pi x}{L}} dx = \frac{1}{2L\Delta} \int_{x_0}^{x_0+\Delta} e^{i \frac{n\pi x}{L}} dx$$

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$$= \frac{1}{2L\Delta} \frac{L}{i\pi} e^{i\pi x \frac{\Delta}{2}} \Big|_{x_0}^{x_0+\Delta} = \frac{1}{2i\pi\Delta} e^{i\pi x_0 \frac{\Delta}{2}} (e^{i\pi \frac{\Delta}{2}} - 1)$$
$$= c_n$$

(equivalently)

$$= \frac{1}{\pi\Delta} e^{i\pi (x_0 + \frac{\Delta}{2}) \frac{\Delta}{2}}$$