HW 08 Solutions

4.4.1 Vibrating string of density ρ_0, tension T_0

(a) Fixed ends: $u(0,t)=u(L,t)=0$

$$u_{tt} = c^2 u_{xx}, \quad c^2 = \frac{T_0}{\rho_0}$$

Natural frequencies are $c\sqrt{\lambda}$ and

$\lambda = \left(\frac{n\pi}{L}\right)^2$ according to section 4.4.

$\Rightarrow$ natural frequencies $= \frac{n\pi c}{L}$, $n=1,2,\dots$

(b) $u(0,t) = \frac{du(H)}{dx} = 0$

$u = \phi(x)\, h(t)$

ODE BVP
$$\begin{cases} \frac{d^2\phi}{dx^2} = -\lambda\phi \\ \phi(0)=0 \\ \frac{d\phi(H)}{dx} = 0 \end{cases}$$

$\Rightarrow \phi = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

$\phi(0) = c_1 = 0$

$\Rightarrow \phi(x) = c_2\sin(\sqrt{\lambda}x)$

$$\frac{d\phi}{dx} = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$$\frac{d\phi(H)}{dx} = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}H) = 0$$

$\lambda<0$, $\lambda=0$ - only trivial sol

$\Rightarrow H\sqrt{\lambda} = -\frac{\pi}{2} + \pi n$

$$\Rightarrow \lambda = \frac{\pi^2(n-\frac{1}{2})^2}{H^2}, \quad n=1,2,\dots$$

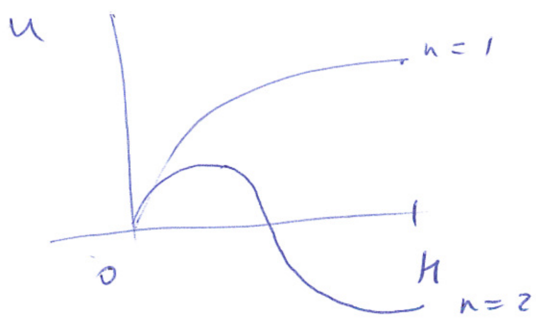
$h'' = -\lambda c^2 h$

$\Rightarrow h = c_1 \cos(c\sqrt{\lambda} t) + c_2 \sin(c\sqrt{\lambda} t)$

$c\sqrt{\lambda}$ – natural frequences

$\Rightarrow c\sqrt{\lambda} = \frac{(n-\frac{1}{2})\pi c}{H}, \quad n = 1, 2, \ldots$

Eigenfunctions $\sin\left(\frac{(n-\frac{1}{2})\pi x}{H}\right)$



(c) Odd harmonics for (a):

$u = \phi(x)\, h(t)$

$H = \frac{L}{2}$

$$\begin{cases} \frac{d^2 h}{dt^2} = -\lambda h \\ \frac{d^2\phi}{dx^2} = -\lambda \phi \\ \phi(0) = \phi(L) = 0 \end{cases}$$

$\lambda = \left(\frac{n\pi}{L}\right)^2$. For n odd: $n = 1, 3, 5, \ldots$

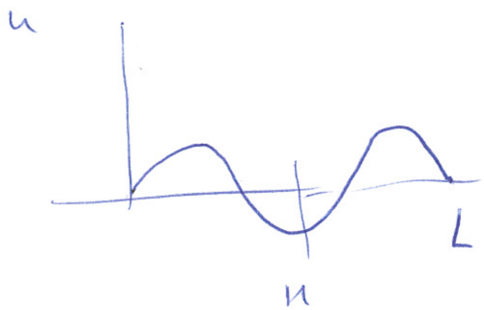
$\lambda = \left(\frac{(2m-1)\pi}{L}\right)^2 = \frac{(m-\frac{1}{2})^2 \pi^2}{(L/2)^2} = \frac{(m-\frac{1}{2})^2 \pi^2}{H^2},$

$m = 1, 2, \ldots$

$\Rightarrow$ frequences of (a) = (b) for $n = 2m-1$, $m = 1, 2, \ldots$

Symmetry: for n odd modes we have $n-1$, i.e. even number of zeros symmetric relative to $x = M$.

$\Rightarrow \frac{du(M)}{dx} = 0.$



4.4.3 $\quad \rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$

(a) β is positive because damping slows down motion. E.g. for $\frac{\partial u}{\partial x} = 0$

$$\Rightarrow \rho_0 \frac{d^2 u}{dt^2} = -\beta \frac{du}{dt}$$

$$\Rightarrow \frac{du}{dt} = \frac{du}{dt}(0)\, e^{-\frac{\beta}{\rho_0} t}$$

– decays for $\beta > 0$ because $\rho_0 > 0$.

(b) BC $\quad u(0,t) = u(L,t) = 0$

IC $\quad u(x,0) = f(x)$ and $\frac{\partial u}{\partial x}(x,0) = g(x)$

Also assume $\beta^2 < 4\pi^2 \frac{\rho_0 T_0}{L^2}$

Separation of variables:

$$u = \phi(x)\, h(t)$$

$$\Rightarrow \frac{\rho_0 \phi \frac{d^2 h}{dt^2} = T_0 h \frac{d^2\phi}{dx^2} - \beta \phi \frac{dh}{dt}}{T_0 \phi h}$$

$$\Rightarrow \frac{\rho_0}{T_0} \frac{\frac{d^2 h}{dt^2}}{h} + \frac{\beta}{T_0} \frac{\frac{dh}{dt}}{h} = \frac{\frac{d^2\phi}{dx^2}}{\phi} = -\lambda$$

ODE BVP: $\begin{cases} \dfrac{d^2\phi}{dx^2} = -\lambda \phi \\ \phi(0) = \phi(L) = 0 \end{cases}$ $\Rightarrow$ $\lambda = \left(\frac{n\pi}{L}\right)^2$, $\phi = \sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, \ldots$

Equation for h:

$\rho_0 h'' + \beta h' + T_0 \lambda h = 0$ — homogeneous linear ODE with constant coefficients

$\Rightarrow$ solve in the form $h = e^{rt}$

$\Rightarrow$ characteristic eq:

$$\rho_0 r^2 + \beta r + \left(\frac{n\pi}{L}\right)^2 T_0 = 0$$

$$\Rightarrow r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 T_0 \left(\frac{n\pi}{L}\right)^2}}{2\rho_0}$$

Since $\beta^2 < 4\rho_0 T_0 \left(\frac{\pi}{L}\right)^2 \Rightarrow$ discriminant $< 0 \ \forall n$

$\Rightarrow$ complex eigenvalue:

$$r_{1,2} = -\frac{\beta}{2\rho_0} \pm i\sqrt{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4\rho_0^2}}$$

General sol:

$$h = c_1 e^{-\frac{\beta}{2\rho_0}t} \cos\left[\sqrt{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4\rho_0^2}}\, t\right]$$

$$+ c_2 e^{-\frac{\beta}{2\rho_0}t} \sin\left[\sqrt{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4\rho_0^2}}\, t\right]$$

Here we also see explanation for (a) that for $\beta < 0$ we would have exp growing solution which is unphysical for damping.

Superposition:

$$u = e^{-\frac{\beta t}{2\rho_0}} \sum_{n=1}^{\infty} \left[a_n \cos(w_n t) + b_n \sin(w_n t)\right] \sin\frac{n\pi x}{L},$$

where $w_n = \sqrt{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4\rho_0^2}}$.

IC : $u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$

$$\Rightarrow \boxed{a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = -\frac{\beta}{2\rho_0} e^{-\frac{\beta t}{2\rho_0}} \sum_{n=1}^{\infty}\left[a_n \cos(w_n t)\right.$$

$$\left.+ b_n \sin(w_n t)\right] \times \sin\left(\frac{n\pi x}{L}\right)\Big|_{t=0} + e^{-\frac{\beta t}{2\rho_0}} \cdot \sum_{n=1}^{\infty} w_n \left[-a_n \sin(w_n t)\right.$$

$$\left. + b_n \cos(w_n t)\right] \sin\left(\frac{n\pi x}{L}\right)\Big|_{t=0}$$

$$= -\frac{\beta}{2\rho_0} \underbrace{\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)}_{= f(x)} + \sum_{n=1}^{\infty} b_n w_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \boxed{b_n w_n = \frac{\beta a_n}{2\rho_0} + \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

already given above

5.3.3

$$H(x) \Big| \quad \frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)]\phi = 0 \quad (*)$$

We need to find $H(x)$ such that the resulting eq is in SL form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\phi = 0 \quad (**)$$

$$H(x)\frac{d^2\phi}{dx^2} + H(x)\alpha(x)\frac{d\phi}{dx} = \frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right]$$

$$= p'\frac{d\phi}{dx} + p\frac{d^2\phi}{dx^2}$$

$\Rightarrow$ terms with $\frac{d\phi}{dx}$, $\frac{d^2\phi}{dx^2}$ $\Big|$ $\left.\begin{array}{l} p'(x) = H(x)\alpha(x) \\ p(x) = H(x) \end{array}\right\} \Rightarrow H' = H\alpha$

$$\frac{dH}{H} = \alpha(x)\,dx$$

$$\Rightarrow H(x) = c_1 e^{\int^x \alpha(t)\,dt}$$

Set e.g. $c_1 = 1 \Rightarrow \boxed{H = e^{\int^x \alpha(t)\,dt}}$

compare $(*)$ and $(**)$: $\Rightarrow$ $p(x) = H(x)$, $q(x) = \gamma H$, $\sigma = \beta H$.

5.3.5 $\phi'' + \lambda\phi = 0$, $\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0$ (8)

(a)

$\Rightarrow \quad \phi = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

$$\frac{d\phi}{dx} = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$\lambda > 0$

$$\frac{d\phi}{dx}(0) = 0 = c_2\sqrt{\lambda} \Rightarrow c_2 = 0$$

$$\cancel{\frac{d\phi}{dx}(L)} = \frac{d\phi}{dx}(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$$

$\lambda = 0 \quad \phi = \text{const.}$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \quad \phi_n = \cos\left(\frac{n\pi}{L}x\right), \quad n = 0, 1, \ldots$$

(a) ∞ of eigenvalues.

(b) $\cos\left(\frac{n\pi x}{L}\right)$ has $n-1$ zeros at $0 < x < L$

(c) Fourier cosine series ~~has~~ use orthogonality of cosines. Completeness - from thm. on Fourier Series.

not assigned

(d) Rayleigh quotient

$$\lambda = \frac{\overbrace{-\phi\frac{d\phi}{dx}\Big|_0^L}^{=0} + \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^L \phi^2 dx} \geq 0 \Rightarrow \lambda < 0 \text{ is not eigenvalue}$$

$\lambda = 0$ is eigenvalue for $\phi = \text{const.}$

5.3.9.c

$$x^2 \frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda\phi = 0$$

$$\phi(1) = 0, \quad \phi(b) = 0$$

Since eq. is equidimensional $\Rightarrow \phi = x^r$

$\Rightarrow \quad r(r-1) + r + \lambda = 0 \Rightarrow r^2 = -\lambda$

$\underline{\lambda > 0}$ $\quad r = \pm i\sqrt{\lambda} \Rightarrow x^r = x^{\pm i\sqrt{\lambda}} = e^{\pm i\sqrt{\lambda}\ln x}$

$\Rightarrow$ general sol $\phi = c_1 \cos(\sqrt{\lambda}\ln x) + c_2 \sin(\sqrt{\lambda}\ln x)$

$\underline{BC}$ $\quad \phi(1) = 0 = c_1$

$\phi(b) = c_2 \sin(\sqrt{\lambda}\ln b) = 0 \Rightarrow \sqrt{\lambda}\ln b = n\pi, \quad n = 1, 2, \ldots$

$$\Rightarrow \lambda = \left(\frac{n\pi}{\ln b}\right)^2$$

Eigen function $\sin\left(\frac{n\pi}{\ln b}\ln x\right)$

$\underline{\lambda = 0}$ $\Rightarrow \quad x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} = x\frac{d}{dx}\left(x\frac{d\phi}{dx}\right) = 0$

$$\Rightarrow \frac{d}{dx}\left(x\frac{d\phi}{dx}\right) = 0$$

$$x \frac{d\varphi}{dx} = c_1$$

$$d\varphi = c_1 \frac{dx}{x}$$

$$\varphi = c_1 \ln x + c_2$$

BC

$$\varphi(1) = 0 = c_2$$

$$\varphi(b) = c_1 \ln b = 0 \Rightarrow c_1 = 0 \Rightarrow \text{not eigenvalue}$$

$\underline{\lambda < 0}$ ~~not assert to analyse.~~ $r = e^{\pm\sqrt{\lambda}\ln x}$

general sol

$$\varphi = c_1 e^{\sqrt{\lambda}\ln x} + c_2 e^{-\sqrt{\lambda}\ln x}$$

$$\varphi(1) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$\varphi(b) = c_1 \underbrace{\left(e^{\sqrt{\lambda}\ln b} - e^{-\sqrt{\lambda}\ln b}\right)}_{\neq 0} = 0$$

$\Rightarrow c_1 = 0$ - only trivial sol

So $\lambda_n = \left(\frac{n\pi}{\ln b}\right)^2 \to \infty$ as $n \to \infty$

Smallest λ: $\lambda_1 = \left(\frac{\pi}{\ln b}\right)^2$