Math 312 HW 6 Solutions

5.5.1 $p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b = 0 \Rightarrow$ self-adjoint

(c) $$\frac{d\phi}{dx}(0) - h\phi(0) = 0, \quad \frac{d\phi}{dx}(L) = 0$$

$\Rightarrow$ both for u and v ($a=0$, $b=L$):

$$\frac{du(0)}{dx} - hu(0) = 0 = \cancel{\frac{d\phi}{dx}}\frac{du}{dx}(L) = 0$$

$$\frac{dv(0)}{dx} - hv(0) = 0 = \frac{dv(L)}{dx}$$

$$\Rightarrow u\left(\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_0^L = u(L)\underbrace{\frac{dv(L)}{dx}}_{=0} - v(0)\underbrace{\frac{du(0)}{dx}}_{=0}$$

$$- u(0)\underbrace{\frac{dv(0)}{dx}}_{=hv(0)} + v(0)\underbrace{\frac{du(0)}{dx}}_{=hu(0)} = -hu(0)v(0) + hv(0)u(0) = 0 \quad \blacksquare$$

(g) $$\phi(L) + \alpha\phi(0) + \beta\frac{d\phi}{dx}(0) = 0$$

$$\frac{d\phi}{dx}(L) + \gamma\phi(0) + \delta\frac{d\phi}{dx}(0) = 0$$

Solving for $u(L)$, $v(L)$; $\frac{du(L)}{dx}$, $\frac{dv(L)}{dx}$

(2)

$$\left.\begin{aligned}
&\underline{u:} \quad u(L) = -\alpha\, u(0) - \beta \frac{du(0)}{dx} \\
&\qquad \frac{du(L)}{dx} = -\gamma\, u(0) - \delta \frac{du(0)}{dx} \\
&\underline{v:} \quad v(L) = -\alpha\, v(0) - \beta \frac{dv(0)}{dx} \\
&\qquad \frac{dv(L)}{dx} = -\gamma\, v(0) - \delta \frac{dv(0)}{dx}
\end{aligned}\right\} \qquad (*)$$

$(a = 0,\ b = L)$

$$\Rightarrow \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) \Big|_0^L$$

$$= u(L) \frac{dv(L)}{dx} - u(0) \frac{dv(0)}{dx} - v(L) \frac{du(L)}{dx} + v(0) \frac{du(0)}{dx}$$

$$= \{ \text{using } (*) \}$$

$$= \left(-\alpha\, u(0) - \beta \frac{du(0)}{dx}\right)\left(-\gamma\, v(0) - \delta \frac{dv(0)}{dx}\right) - u(0) \frac{dv(0)}{dx}$$

$$- \left(-\alpha\, v(0) - \beta \frac{dv(0)}{dx}\right)\left(-\gamma\, u(0) - \delta \frac{du(0)}{dx}\right) + v(0) \frac{du(0)}{dx}$$

$$= \alpha\gamma\, u(0)v(0) + \beta\gamma \frac{du(0)}{dx} v(0) + \alpha\delta\, u(0)\frac{dv(0)}{dx} + \beta\delta \frac{du(0)}{dx}\frac{dv(0)}{dx}$$

$$- u(0)\frac{dv(0)}{dx}$$

$$- \alpha\gamma\, v(0)u(0) - \beta\gamma \frac{dv(0)}{dx} u(0) - \alpha\delta\, v(0)\frac{du(0)}{dx} - \beta\delta \frac{dv(0)}{dx}\frac{du(0)}{dx}$$

$$+ v(0)\frac{du(0)}{dx}$$

$$= \frac{du(0)}{dx} v(0)\left[\beta\gamma - \alpha\delta + 1\right] + u(0)\frac{dv(0)}{dx}\left[\alpha\delta - 1 - \beta\gamma\right] = 0$$

$$\Rightarrow \boxed{\alpha\delta - \beta\gamma - 1 = 0}$$

5.5.5 $\quad L = \frac{d^2}{dx^2} + 6\frac{d}{dx} + 9$

(a)

$$L(e^{rx}) = r^2 e^{rx} + r\,6\,e^{rx} + 9e^{rx}$$

$$= (r+3)^2 e^{rx}$$

(b) $L(y) = 0$ using part (a): $r = -3$ double root

$$\Rightarrow y = (c_1 + c_2 x)e^{-3x}$$

5.5A.5a) Use example of p. 181

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix} \qquad \det(A - \lambda I) = \lambda^2 - (-2)\lambda - 9 = 0$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{40}}{2} = -1 \pm \sqrt{10} \ \text{- real}$$

$\underline{\lambda_1 = -1 + \sqrt{10}}$

$$(A - \lambda I)\vec{v} = \begin{pmatrix} 3-\sqrt{10} & 1 \\ 1 & -3-\sqrt{10} \end{pmatrix} \vec{v} = 0 \quad , \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} ,$$

$$\Rightarrow (3 - \sqrt{10}) v_1 + v_2 = 0$$

Take $v_1 = 1 \Rightarrow v_2 = \sqrt{10} - 3$, $\vec{v} = \begin{pmatrix} 1 \\ \sqrt{10} - 3 \end{pmatrix}$

$\underline{\lambda_2 = -1 - \sqrt{10}}$

$$(A - \lambda I)\vec{u} = \begin{pmatrix} 3+\sqrt{10} & 1 \\ 1 & -3+\sqrt{10} \end{pmatrix} \vec{u} \qquad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

~~Take~~ $\Rightarrow (3 + \sqrt{10}) u_1 + u_2 = 0$

Take $u_1 = 1 \Rightarrow u_2 = -3 - \sqrt{10}$ $\quad \vec{u} = \begin{pmatrix} 1 \\ -3-\sqrt{10} \end{pmatrix}$

$$\vec{v} \cdot \vec{u} = 1 \cdot 1 + (\sqrt{10} - 3)(-3 - \sqrt{10}) = 1 + (-10 + 9) = 0$$

$$\Rightarrow \boxed{\vec{v} \perp \vec{u}}.$$

5.6.1.c

$$\begin{cases} \frac{d^2\Phi}{dx^2} + \lambda\Phi = 0 \\ \Phi(0) = 0 \\ \frac{d\Phi}{dx}(1) + \Phi(1) = 0 \end{cases}$$

Rayleigh quotient : $\lambda = \dfrac{-p\Phi\frac{d\Phi}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\Phi}{dx}\right)^2 - q\Phi^2\right]dx}{\int_a^b \Phi^2\sigma\,dx}$

In our case $p=1,\ \sigma=1,\ a=0,\ b=1,\ q=0$

$$-\Phi\frac{d\Phi}{dx}\Big|_0^1 = -\Phi(1)\underbrace{\frac{d\Phi(1)}{dx}}_{=-\Phi(1)} + \underbrace{\Phi(0)}_{=0}\frac{d\Phi(0)}{dx} = \Phi^2(1)$$

$$\Rightarrow \lambda = \frac{\Phi^2(1) + \int_0^1 \left(\frac{d\Phi}{dx}\right)^2 dx}{\int_0^1 \Phi^2\,dx}$$

For trial function u_T according to 5.6.6 of textbook

$$\lambda_1 \le \frac{u_T^2(1) + \int_0^1 \left(\frac{du_T}{dx}\right)^2 dx}{\int_0^1 u_T^2\,dx}$$

trial function should satisfy BC (6)

and be continuous with no zeros.

Look for trial function as a parabola:

$$u_T = ax + bx^2 + c, \quad \text{here } c=0 \text{ because } u_T(0)=0$$

Condition at $x=1$: $\frac{du_T}{dx} = a + 2bx$

$$\frac{du_T}{dx}(1) + u_T(1) = a + 2b + a + b = 0$$

$$\Rightarrow \quad b = -\frac{2}{3}a$$

We can choose different values of a,

but optimal is to find min as a function of a:

$$\frac{du_T}{dx} = a + 2bx = a - \frac{4}{3}ax$$

$$u_T = a\left(x - \frac{2}{3}x^2\right)$$

$$\frac{u_T^2(1) + \int_0^1 \left(\frac{du_T}{dx}\right)^2 dx}{\int_0^1 u_T^2\,dx} = \frac{\frac{1}{9}a^2 + a^2\int_0^1 \left(1-\frac{4}{3}x\right)^2 dx}{a^2\int_0^1 \left(x-\frac{2}{3}x^2\right)^2 dx}$$

$$= \frac{\frac{1}{9} + \int_0^1 \left(1 - \frac{8}{3}x + \frac{16}{9}x^2\right)dx}{\int_0^1 \left(x^2 - \frac{4}{3}x^3 + \frac{4}{9}x^4\right)dx}$$

$$= \frac{\frac{1}{9} + \left(x - \frac{4}{3}x^2 + \frac{16}{27}x^3\right)\Big|_0^1}{\frac{x^3}{3} - \frac{x^4}{3} + \frac{4}{45}x^5\Big|_0^1} = \frac{\frac{1}{9} + \left(1 - \frac{4}{3} + \frac{16}{27}\right)}{\frac{1}{3} - \frac{1}{3} + \frac{4}{45}}$$

$$= \frac{\frac{1}{9} + \frac{27 - 36 + 16}{27}}{\frac{4}{45}} = \frac{\frac{3+7}{27}}{\frac{4}{45}} = \frac{10}{27} \cdot \frac{45}{4} = \boxed{\frac{25}{6} = 4\frac{1}{6}}$$

– independent on a.

5.7.1

$u(0) = u(1) = 0$ - fixed ends

$$c(x)^2 = 1 + 4\alpha^2 \left(x - \frac{1}{2}\right)^2$$

From eqs on p 192:

$$\frac{\pi}{L} \underbrace{\sqrt{\frac{T_0}{\rho_{max}}}}_{= c_{min}} \le \sqrt{\lambda_1} \le \frac{\pi}{L} \underbrace{\sqrt{\frac{T_0}{\rho_{min}}}}_{= c_{max}^2} \qquad L = 1$$

$$c_{min}^2 = c^2\left(\frac{1}{2}\right) = 1$$

$$c_{max}^2 = c^2(0) = 1 + \alpha^2$$

$$\Rightarrow \frac{\pi}{2} \le \sqrt{\lambda_1} \le \pi \sqrt{1 + \alpha^2}$$

The circular frequency is $\sqrt{\lambda_1}$ and actual frequency is $\frac{\sqrt{\lambda_1}}{2\pi}$

$$\Rightarrow$$

$$\frac{1}{2} \le \frac{\sqrt{\lambda_1}}{2\pi} \le \frac{1}{2} \sqrt{1 + \alpha^2}$$