HW 07 Solutions

Math 3?2



5.3.3 $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$, $\frac{d\phi}{dx}(0) = 0$

(a) $\frac{d\phi}{dx}(L) + h\,\phi(L) = 0$,

$h > 0$

$$\lambda = \frac{-p\phi\frac{d\phi}{dx}\Big|_0^L + \int_0^L \left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right]dx}{\int_0^L \phi^2 \sigma\, dx}$$

$$= \left\{ \begin{array}{l} p=1,\ \sigma=1,\ q=0 \\ \frac{d\phi}{dx}(L) = -h\phi(L) \end{array} \right\}$$

$$= \frac{h\,\phi(L)^2 + \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^L \phi^2\, dx} \geq 0 \quad \text{because } h > 0$$

Also iff to have $= 0$ requires $\phi(L) = 0$, $\frac{d\phi}{dx} = 0$

$\Rightarrow \phi(x) \equiv 0$ - only

trivial sol $\Rightarrow$ $\boxed{\lambda > 0}$

(b) By (a) $\lambda > 0$

$\Rightarrow \phi = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

$$\frac{d\Phi}{dx} = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$$\frac{d\Phi}{dx}(0) = 0 = c_2\sqrt{\lambda} \Rightarrow c_2 = 0$$

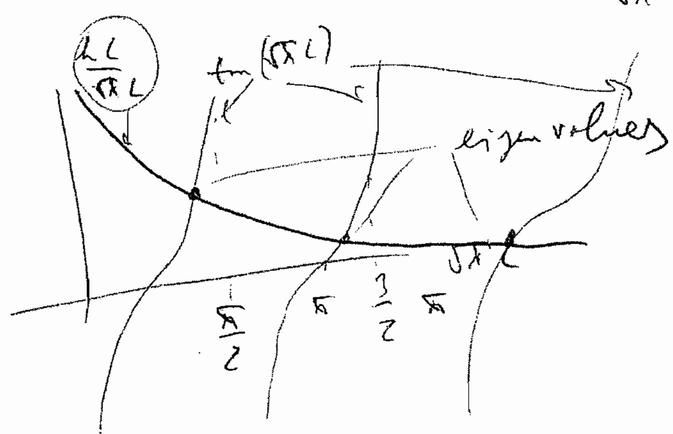
$$\Phi(x) = c_1\cos(\sqrt{\lambda}x)$$

BC at $x = L$: $\displaystyle \frac{d\Phi}{dx}(L) + h\,\Phi(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L)$

$$+ h\,c_1\cos(\sqrt{\lambda}L) = 0$$

$$\Rightarrow \tan(\sqrt{\lambda}L) = \frac{h}{\sqrt{\lambda}} = \frac{hL}{\sqrt{\lambda}L}$$

$\Rightarrow$




$\frac{hL}{\sqrt{\lambda}L}$ = hyperbola in $\sqrt{\lambda}L$

$$0 < \sqrt{\lambda_1}L < \frac{\pi}{2}$$

$$\pi < \sqrt{\lambda_2}L < \frac{3}{2}\pi$$

$$2\pi < \sqrt{\lambda_3}L < \frac{5}{2}\pi$$

$$\Rightarrow \boxed{(n-1)\pi < \sqrt{\lambda_n}\,L < \left(n-\tfrac{1}{2}\right)\pi}$$

$n = 1, 2, \ldots$

As $n \to \infty$ $\Rightarrow \lambda_n \to \left(\frac{\pi}{L}(n-1)\right)^2$

Approach lower ~~band~~ bound

5.8.7

$$\frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\varphi(0) = 0, \quad \varphi(\pi) = 2\frac{d\varphi}{dx}(0)$$

(a) $$\int_0^\pi \left(u\frac{d^2v}{dx^2} - v\frac{d^2u}{dx^2}\right)dx$$

$$= u\frac{dv}{dx} - v\frac{du}{dx}\Big|_0^\pi - \underbrace{\int_0^\pi\left(\frac{du}{dx}\frac{dv}{dx} - \frac{dv}{dx}\frac{du}{dx}\right)dx}_{=0}$$

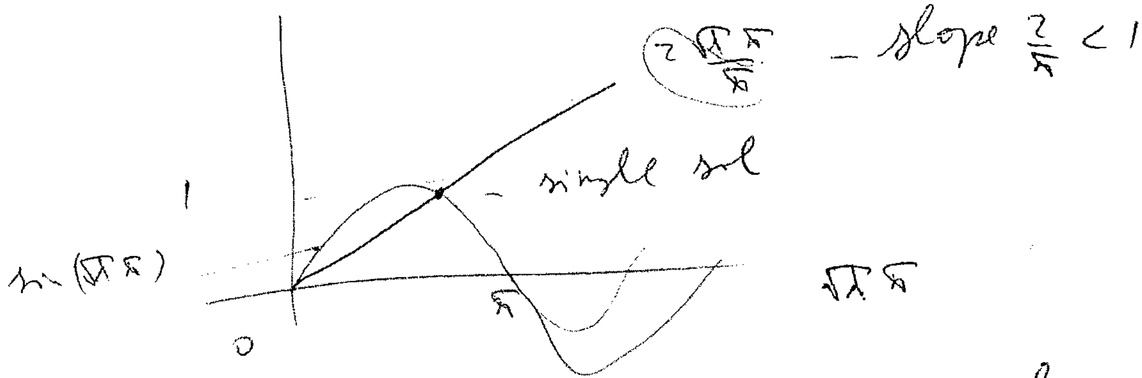
$$= u(\pi)\frac{dv(\pi)}{dx} - v(\pi)\frac{du(\pi)}{dx} = \underbrace{2\frac{du(0)}{dx}\cdot\frac{dv(\pi)}{dx} - 2\frac{dv(0)}{dx}\frac{du(\pi)}{dx}}_{\text{generally } \neq 0}$$

(for general values of $\frac{du(0)}{dx}$, $\frac{dv(0)}{dx}$, $\frac{dv(\pi)}{dx}$, $\frac{du(\pi)}{dx}$.

(b) If $\lambda > 0 \Rightarrow \varphi(0) = 0$ results in $\varphi = \sin(\sqrt{\lambda}\,x)$

$$\varphi(\pi) - 2\frac{\varphi(0)}{dx} = \sin(\sqrt{\lambda}\,\pi) - 2\sqrt{\lambda}\underbrace{\cos 0}_{=1} = \sin(\sqrt{\lambda}\,\pi) - 2\sqrt{\lambda} = 0$$

Graphical sol:



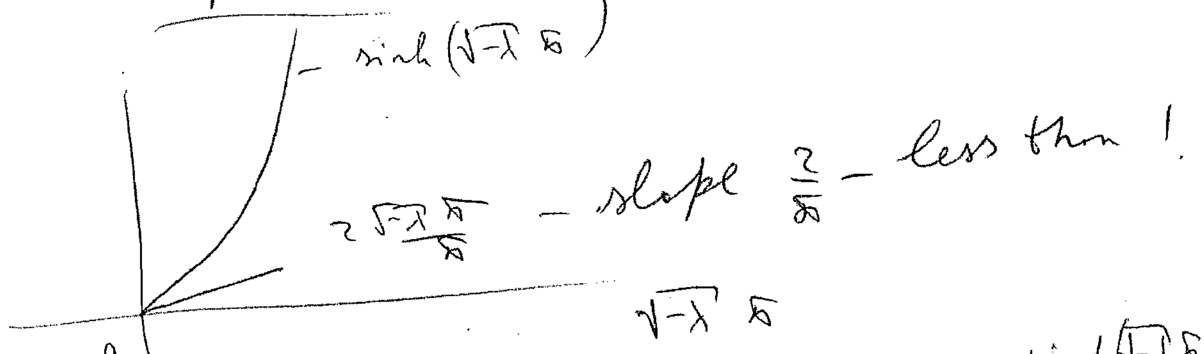

$\Rightarrow$ 1 positive eigenvalue $< 1$.

(c) If $\lambda < 0 \Rightarrow \phi(0) = 0$

gives $\phi = \sinh(\sqrt{-\lambda}\, x)$

$$\phi(\pi) - 2\frac{d\phi(0)}{dx} = \sinh(\sqrt{-\lambda}\,\pi) - 2\sqrt{-\lambda} = 0$$

Graphical sol.




$\Rightarrow$ no solution of $\sinh(\sqrt{-\lambda}\,\pi) = 2\sqrt{-\lambda}$ for $\lambda < 0$.

(d) $\lambda = 0 \Rightarrow \phi = c_1 x + c_2$

$\phi(0) = c_2 = 0$

$\Rightarrow \phi(x) = c_1 x$

$\phi(\pi) - 2\frac{d\phi(0)}{dx} = c_1 \pi - 2c_1 = 0$

$\Rightarrow c_1 = 0$ - only trivial sol

$\Rightarrow \lambda = 0$ is not eigenvalue.

(e) Since BC. are not in the form eq 5.3.2 of book, then this is not a regular Sturm-Liouville eigenvalue problem $\Rightarrow$ can have complex eigenvalues.