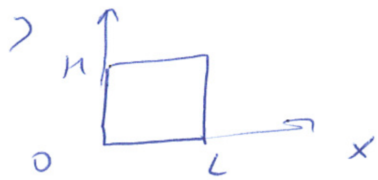
math 312 HW Solutions HW 8

7.3.1



(a)

$$u_t = k(u_{xx}+u_{yy})$$

IC: $u(x,y,0)=\alpha(x,y)$

$$u = \phi(x,y)\,h(t) = f(x)\,g(y)\,h(t)$$

$$\frac{dh}{dt} = -\lambda k h$$

$$\nabla^2\phi + \lambda\phi = 0 \Rightarrow \frac{d^2 f}{dx^2} = -\mu f$$

$$\frac{d^2 g}{dy^2} + (\lambda-\mu)g = 0$$

(a) $u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,H,t) = 0$

$$\Rightarrow \begin{cases} \dfrac{d^2 f}{dx^2} = -\mu f \\ f(0) = f(L) = 0 \end{cases} \Rightarrow \mu = \left(\frac{n\pi}{L}\right)^2, \; n = 1, 2, \ldots$$

Eigenfunctions: $f_n = \sin\left(\frac{n\pi x}{L}\right)$

$$\begin{cases} \dfrac{d^2 g}{dy^2} = -(\lambda - \mu) g \\ g(0) = g(H) = 0 \end{cases} \quad \Rightarrow \lambda_{nm} - \mu = \left(\frac{m\pi}{H}\right)^2$$

Eigenfunction:

$$g_{nm} = \sin \frac{m\pi y}{H}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

$$\varphi = \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}, \quad n, m = 1, 2, \ldots$$

Superposition:

$$\boxed{u = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H} e^{-\lambda_{nm} k t}}$$

I.C. $f = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{L}$

From orthogonality:

$$\boxed{a_{nm} = \frac{4}{2H} \int_0^H \int_0^L f \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right) dx\, dy}$$

As $t \to \infty \Rightarrow u \to 0$ (because all $\lambda > 0$)

the only equilibrium temperature with $u|_\Gamma = 0$.

c $\frac{\partial u}{\partial x}(0,y,t) = \frac{\partial}{\partial x}(L,y,t) = 0$

$u(x,0,t) = 0$, $u(x,H,t) = 0$

$u = \phi(x,y)\,h(t)$

$\frac{dh}{dt} = -\lambda k h$

$\nabla^2 \phi = -\lambda \phi$

Separ. of variables: $\phi = f(x)\,g(y)$

$$\Rightarrow \begin{cases} \frac{d^2 f}{dx^2} = -\mu f \\ f'(0) = f'(L) = 0 \end{cases} \Rightarrow \mu = \left(\frac{n\pi}{L}\right)^2$$

Eigenfunction:

$f = \cos\frac{n\pi x}{L}$, $n = 0, 1, \ldots$

$$\begin{cases} \frac{d^2 g}{dy^2} = -(\lambda - \mu) g \\ g(0) = g(H) = 0 \end{cases} \Rightarrow \lambda - \mu = \left(\frac{m\pi}{H}\right)^2, \; m = 1, 2, \ldots$$

$$\Rightarrow \lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

Eigenfunction:

$$\phi = \cos\frac{n\pi x}{L}\sin\frac{m\pi y}{H}, \quad n = 0, 1, \ldots$$
$$m = 1, 2, \ldots$$

Superposition:

$$u = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{nm}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)e^{-\lambda_{nm}kt}$$

$$\text{IC}: \quad f = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{nm}\cos\left(\frac{n\pi x}{L}\right)\sin\frac{m\pi y}{H}$$

By orthogonality:

$$a_{nm} = \frac{\int_0^H\int_0^L f\cos\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,dx\,dy}{\int_0^H\int_0^L \cos^2\left(\frac{n\pi x}{L}\right)\sin^2\left(\frac{m\pi y}{H}\right)dx\,dy}$$

here $$\boxed{\int_0^H\int_0^L \cos^2\left(\frac{n\pi x}{L}\right)\sin^2\left(\frac{m\pi y}{H}\right)dx\,dy = \begin{cases}\frac{1}{2}LH, & n = 0\\ \frac{1}{4}LH, & n \geq 1\end{cases}}$$

As $t \to \infty \Rightarrow u \to 0$ because zero BC for u at $y = 0$ and $y = H$ implies that $u \equiv 0$ ~~for~~ ~~for~~ because $\lambda > 0$.

7.3.2 b

$$u_t = k(u_{xx} + u_{yy} + u_{zz})$$

$0 < x < L$

$0 < y < H$

$0 < z < W$

$\frac{\partial u}{\partial x}(0,y,z,t) = \frac{\partial u}{\partial x}(L,y,z,t)$

$= \frac{\partial u}{\partial y}(x,0,z,t) = \frac{\partial u}{\partial y}(x,H,z,t) =$

$\frac{\partial u}{\partial z}(x,y,0,t) = \frac{\partial u}{\partial z}(x,y,W,t)$

$= 0$

$u(x,y,z,0) = \alpha(x,y,z)$

$u(x,y,z,t) = \phi(x,y,z)\, h(t)$

$\Rightarrow \frac{dh}{dt} = -\lambda k h$

$\nabla^2 \phi = -\lambda \phi$

Extra separation of variables:

$$\phi(x,y) = f(x)\, g(y)\, Q(z)$$

$$\nabla^2 \phi + \lambda \phi = f_{xx}\, g Q + f\, g_{yy} Q + f g Q_{zz} + \lambda f g Q = 0$$

$$\Rightarrow \underbrace{\frac{f_{xx}}{f}}_{= -\mu} + \underbrace{\frac{g_{yy}}{g}}_{= -\nu} + \underbrace{\frac{Q_{zz}}{Q}}_{= \mu + \nu} + \lambda = 0$$

$$\Rightarrow \begin{cases} \dfrac{d^2 f}{dx^2} = -\mu f \\ f'(0) = f'(L) = 0 \end{cases} \Rightarrow \mu = \left(\frac{n\pi}{L}\right)^2$$

$$f = \cos\frac{n\pi x}{L}, \quad n = 0, 1, \ldots$$

$$\begin{cases} \dfrac{d^2 g}{dy^2} = -\nu g \\ g'(0) = g'(H) = 0 \end{cases} \Rightarrow \nu = \frac{m\pi}{H}$$

$$g = \cos\frac{m\pi y}{H}, \quad m = 0, 1, \ldots$$

$$\begin{cases} \dfrac{d^2 Q}{dz^2} = -(\lambda - \mu - \nu) Q \\ Q'(0) = Q'(W) = 0 \end{cases} \Rightarrow \lambda - \mu - \nu = \left(\frac{\ell\pi}{W}\right)^2$$

$$Q = \cos\left(\frac{\ell\pi z}{W}\right), \quad \ell = 0, 1, 2, \ldots$$

$$\Rightarrow \lambda_{nm\ell} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{\ell\pi}{W}\right)^2$$

$$\phi = \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos\left(\frac{\ell\pi z}{W}\right)$$

$$n, m, \ell = 0, 1, 2, \ldots$$

Superposition:

$$\boxed{u = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} a_{nml} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{L}\right)\cos\left(\frac{l\pi z}{W}\right) e^{-\lambda_{nml} kt}}$$

$$IC: \quad f = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} a_{nml} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{L}\right)\cos\left(\frac{l\pi z}{W}\right)$$

$\Rightarrow$ by orthogonality:

$\alpha(x,y,z)$

$$\boxed{a_{nml} = \frac{\int_0^W\int_0^H\int_0^L \cancel{f} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos\left(\frac{l\pi z}{W}\right) dx\,dy\,dz}{\int_0^W\int_0^H\int_0^L \cancel{f} \cos^2\left(\frac{n\pi x}{L}\right)\cos^2\left(\frac{m\pi y}{L}\right)\cos^2\frac{l\pi z}{W}\, dx\,dy\,dz}}$$

As $t \to \infty$ all terms will decay but

$$A_{000} = \frac{1}{LHW}\int_0^L\int_0^H\int_0^W \cancel{\alpha(x,y)}\, f \, dx\,dy\,dz$$

$$\Rightarrow u(x,y,z,t)\Big|_{t\to\infty} = A_{000}$$

7.3.4.b

$$u_{tt} = c^2(u_{xx} + u_{yy})$$

$$0 < x < L, \quad 0 < y < H$$

IC: $u(x,y,0) = 0 \qquad \frac{\partial u}{\partial t}(x,y,0) = \alpha(x,y)$

BC: $\frac{\partial u}{\partial x}(0,y,t) = \frac{\partial u}{\partial x}(L,y,t) = 0$

$$\frac{\partial u}{\partial y}(x,0,t) = \frac{\partial u}{\partial y}(x,H,t) = 0$$

$$u = \phi(x,y) \, h(t)$$

$$\frac{d^2h}{dt^2} = -\lambda c^2 h. \Rightarrow h = c_1 \cos(\sqrt{\lambda} t) + c_2 \sin(\sqrt{\lambda} t)$$

$h(0) = 0 \Rightarrow \boxed{c_1 = 0}$ if $\lambda > 0$

and $h = c_3 t$ if $\lambda = 0$

Separation of variables for ϕ:

$$\phi = f(x) \, g(y)$$

$$\Rightarrow \begin{cases} \dfrac{d^2f}{dx^2} = -\mu f \\ f'(0) = f'(L) = 0 \end{cases} \Rightarrow \mu = \left(\frac{n\pi}{L}\right)^2$$

Eigenfunction

$$f = \cos\frac{n\pi x}{L}, \quad n = 0, 1, \ldots$$

$$\begin{cases} \dfrac{d^2 g}{dy^2} = -(\lambda - \mu) g \\ g'(0) = g'(H) = 0 \end{cases} \Rightarrow \lambda - \mu = \left(\frac{m\pi}{H}\right)^2$$

Eigenfunction

$$g = \cos\left(\frac{m\pi y}{H}\right), \quad m = 0, 1, \ldots$$

$$\Rightarrow \lambda_{nm} = \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}$$

Superposition:

$$\boxed{u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{nm} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) h_{nm}(t)}$$

Based of $h(0)=0$:

$$\boxed{h_{nm}(t) = \begin{cases} t, & n = m = 0 \\ \sin(\omega_{nm} t), & \text{otherwise}, \end{cases}}$$

here $\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}$

Secon IC: $$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \alpha(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{nm} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) \cdot h'_{nm}(0)$$

$$\Rightarrow A_{nm} h'_{nm}(0) = \frac{\int_0^H \int_0^L f \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) dx\, dy}{\int_0^H \int_0^L \cos^2\left(\frac{n\pi x}{L}\right) \cos^2\left(\frac{m\pi y}{L}\right) dx\, dy}.$$

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Note that $h'_{nm}(0) = 1$ for $n = m = 0$

but otherwise $h'_{nm}(0) = \omega_{nm}$

7.3.7 c

$$u_{xx} + u_{yy} + u_{zz} = 0$$

$$0 < x < L, \quad 0 < y < W, \quad 0 < z < H$$

$$\frac{\partial u}{\partial x}(0, y, z) = 0$$

$$\frac{\partial u}{\partial x}(L, y, z) = \alpha(x, y, z)$$

$$\frac{\partial u}{\partial y}(x, 0, z) = 0$$

$$\frac{\partial u}{\partial y}(x, W, z) = 0$$

$$\frac{\partial u}{\partial z}(x, y, 0) = \frac{\partial u}{\partial z}(x, y, H) = 0$$

$u = h(x)\,\Phi(y,z) \Rightarrow$

$\dfrac{d^2 h}{dx^2} = \beta h$ and $\nabla^2\Phi + \beta\Phi = 0$

$\dfrac{\partial\Phi}{\partial y}(0,z) = \dfrac{\partial\Phi}{\partial y}(W,z) = 0$

$\dfrac{\partial\Phi}{\partial z}(y,0) = \dfrac{\partial\Phi}{\partial z}(y,H) = 0$

$\Rightarrow h = \cosh(\sqrt{\beta}\,x)$

because $h'(0) = 0$

sol. is: $\phi = f(y)\,g(z)$

From 7.3.7.(b):

$$\begin{cases} \dfrac{d^2 f}{dy^2} = -\mu f \\ f'(0) = f'(W) = 0 \end{cases} \qquad \mu = \left(\frac{n\pi}{W}\right)^2$$

Eigenfunction:

$f = \cos\dfrac{n\pi y}{W}, \quad n = 0, 1, \ldots$

$$\begin{cases} \dfrac{d^2 g}{dz^2} = -(\lambda - \mu)\,g \\ g'(0) = g'(H) = 0 \end{cases} \Rightarrow \lambda - \mu = \left(\frac{m\pi}{H}\right)^2$$

Eigenfunction

$g = \cos\left(\dfrac{m\pi z}{H}\right), \quad m = 0, 1, \ldots$

$$\boxed{\beta_{nm}^2 = \left(\frac{n\pi}{W}\right)^2 + \left(\frac{m\pi}{H}\right)^2}$$

Superposition:

$$\boxed{u(x,y,z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{nm}\cos\left(\frac{n\pi y}{W}\right)\cos\left(\frac{m\pi z}{H}\right)\cosh(\beta_{nm}x)}$$

Nonhomogeneous boundary condition:

$$\left.\frac{\partial u}{\partial x}\right|_{x=L}=\alpha(y,z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{nm}\beta_{nm}\cancel{\sinh\left(\frac{n\pi y}{W}\right)\cos\left(\frac{m\pi z}{H}\right)}$$

$$\times\ \sinh(\beta_{nm}L)\cos\left(\frac{n\pi y}{W}\right)\cos\left(\frac{m\pi z}{H}\right)$$

~~At~~

$\Rightarrow$ orthogonality:

$$\boxed{A_{nm}\sinh(\beta_{nm}L)=\frac{\int_0^H\int_0^W\alpha(y,z)\cos\left(\frac{n\pi y}{W}\right)\cos\left(\frac{m\pi z}{H}\right)dy\,dz}{\int_0^H\int_0^W\cos^2\frac{n\pi y}{W}\cos^2\frac{m\pi z}{H}\,dy\,dz}}$$

At $n=m=0$: $\sinh(0\,L)=0$

$\Rightarrow$ require $\boxed{\int_0^H\int_0^W\alpha(y,z)\,dy\,dz=0}$

$\Rightarrow$ A_{00} - arbitrary.

7.5.6

$$I = \iint_R \left[u L(v) - v L(u) \right] dx\, dy$$

$$L = \nabla^2 + q(x,y)$$

$$L(u) = \nabla^2 u + q u$$

$$L(v) = \nabla^2 v + q v$$

$$\Rightarrow \quad u L(v) - v L(u) = u \nabla^2 v + \cancel{q u v} - v \nabla^2 u - \cancel{q v u}$$

$$= u \nabla^2 v - v \nabla^2 u$$

$$I = \iint_R \left[u L(v) - v L(u) \right] dx\, dy = \iint_R (u \nabla^2 v - v \nabla^2 u)\, dx\, dy$$

$$= \oint_{\Gamma} (u \vec{\nabla} v - v \vec{\nabla} u) \cdot \vec{n}\, ds$$

boundary of R