$r=1$ (quarter disk) $0<\theta<\frac{\pi}{2}$

7.7.3a Note typo in the book: they are talking about a vibrating membrane but eq. for $h(t)$ is from heat. It must be replaced by $\frac{d^2h}{dt^2}=-\lambda c^2 h$.

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$u(r,\theta,t) = \phi(x,y)\, h(t)$$

$\Rightarrow \nabla^2\phi+\lambda\phi=0$, $\frac{d^2h}{dt^2}=-\lambda c^2 h$

$\Rightarrow \phi = f(r)\, g(\theta)$, $\Rightarrow h(t)=c_1\cos(c\sqrt{\lambda}\,t)+c_2\sin(c\sqrt{\lambda}\,t)$

$$\Rightarrow \frac{d^2g}{d\theta^2}=-\mu g$$

and $$r\frac{d}{dr}\left(r\frac{df}{dr}\right)+(\lambda r^2-\mu)f=0$$

ODE BVP
$$\begin{cases}\dfrac{d^2g}{d\theta^2}=-\mu g\\[4pt] g(0)=g\left(\frac{\pi}{2}\right)=0\end{cases}$$
- standard eigenvalue problem with $L=\frac{\pi}{2}$

$$\Rightarrow \mu=\left(\frac{m\pi}{L}\right)^2=4m^2$$

eigenfunctions: $\sin(2m\theta)$ $m=1,2,3\ldots$

Second

ODE BVP:

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right)+(\lambda r^2-\overset{4m^2}{\overset{\|}{\mu}})f=0$$

$$\Rightarrow r^2f''+rf'+(\lambda r^2-4m^2)f=0$$

$$\Rightarrow f=C_1J_{2m}(\sqrt{\lambda}a)+\underset{\underset{0}{\|}}{C_2}Y_{2m}(\sqrt{\lambda}a)=0$$

by singularity condition $|f(0)|<\infty$

$$\Rightarrow J_{2m}(\sqrt{\lambda}a)=0$$

$$\Rightarrow \lambda_{mn}=\left(\frac{z_{mn}}{a}\right)^2,\quad n=1,2,\dots \quad m=1,2,\dots$$

z_{mn} - nth zero of J_{2m}

Circular frequencies $\omega_{mn}=c\sqrt{\lambda_{mn}}=\frac{cz_{mn}}{a}$

7.7.8 (a)

$$\frac{\partial u}{\partial t}=k\nabla^2u,\quad u(r,\theta,0)=f(r,\theta)$$

$$\frac{\partial u}{\partial r}(a,\theta,t)=0 \quad \text{- insulated boundary.}$$

Separation of variables:

$u(r,\theta,t)=f(r)g(\theta)h(t)$ - as in Section 7.7.

$$\Rightarrow \frac{dh}{dt}=-k\lambda h\Rightarrow h(t)=ce^{-k\lambda t}$$

ODE BVP

$$\frac{d^2 g}{d\theta^2} + \mu g(\theta) = 0 , \quad g(0) = g(2\pi)$$

$$\Rightarrow g(\theta) = c_1 \cos(m\theta) + c_2 \sin(m\theta), \quad m = 0, 1, \ldots$$

Second ODE BVP:

$$r^2 \frac{d^2 f}{dr^2} + r \frac{df}{dr} + (\lambda r^2 - m^2) f = 0$$

$|f(0)| < \infty \Rightarrow$ Eigenfunctions $J_m(\sqrt{\lambda}\, r)$

$\frac{df(a)}{dr} = 0 = \sqrt{\lambda}\, J_m'(\sqrt{\lambda}\, a) = 0$

$\Rightarrow$ (a) $\lambda = 0 \Rightarrow$ nontrivial sol $J_m(0)$ only for $m=0$

$\Rightarrow$ const - eigenfunction

(b) $\lambda > 0 \Rightarrow J'(\sqrt{\lambda}\, a) = 0$, $\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$,

z_{mn} - nth zero of $J_m'(z)$

Rayleigh quotient $\lambda = \dfrac{-r^2 f \frac{df}{dr}\Big|_0^a + \int_0^a \left(r \left(\frac{df}{dr}\right)^2 + \frac{m^2}{r} f^2 \right) dr}{\int_0^a f^2 r\, dr} \geq 0$

$p = r,\ \sigma = r,$
$q = -\frac{m^2}{r}$

$\Rightarrow \lambda < 0$ is excluded. (not eigenvalue)

Superposition:

$$\Rightarrow u(r, \theta, t) = A_0 + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} A_{mn} \cos(m\theta)\, J_m(\sqrt{\lambda_{mn}}\, r)\, e^{-k\lambda_{mn} t}$$

from $\lambda = 0$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \sin(m\theta)\, J_m(\sqrt{\lambda_{mn}}\, r)\, e^{-k\lambda_{mn} t}$$

IC: $u(r,\theta,0) = f(r,\theta) =$

$$= A_0 + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} A_{mn} \cos(m\theta)\, J_m(\sqrt{\lambda_{mn}}\, r)$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \sin(m\theta)\, J_m(\sqrt{\lambda_{mn}}\, r)$$

By 2D orthogonality:

$$A_{mn} = \frac{\int_{-\pi}^{\pi} d\theta \int_0^a r\,dr\, f(r,\theta) \cos(m\theta)\, J_m(\sqrt{\lambda_{mn}}\, r)}{\int_{-\pi}^{\pi} d\theta \int_0^a r\,dr\, \cos^2(m\theta)\, J_m^2(\sqrt{\lambda_{mn}}\, r)}$$

$m = 0, 1, \ldots$
$n = 1, 2, \ldots$

$$B_{mn} = \frac{\int_{-\pi}^{\pi} d\theta \int_0^a r\,dr\, f(r,\theta) \sin(m\theta)\, J_m(\sqrt{\lambda_{mn}}\, r)}{\int_{-\pi}^{\pi} d\theta \int_0^a r\,dr\, \sin^2(m\theta)\, J_m^2(\sqrt{\lambda_{mn}}\, r)}.$$

$m = 1, 2, \ldots$
$n = 1, 2, \ldots$

And $$A_0 = \frac{\int_{-\pi}^{\pi} d\theta \int_0^a r\,dr\, f(r,\theta)}{\int_{-\pi}^{\pi} d\theta \int_0^a r\,dr}$$

As $t \to \infty \Rightarrow u(r,\theta,t) \to A_0$

7.7.9.6

$$\frac{\partial u}{\partial t} = K\nabla^2 u \qquad 0<\theta<\pi$$

$$u(r,\theta,0) = f(r,\theta)$$

BC $\frac{\partial u}{\partial \theta}(r,0,t) = \frac{\partial u}{\partial \theta}(r,\pi,t) = 0$

$$\frac{\partial u}{\partial r}(a,\theta,t) = 0$$

$$u = f(r)\, g(\theta)\, h(t)$$

Similar to Section 7.7 but different for h:

$$\frac{dh}{dt} = -\lambda k h$$

ODE BVP $\begin{cases} \frac{d^2 g}{d\theta^2} = -\mu g \\ g'(0) = g(\pi) = 0 \end{cases}$ $\Rightarrow$ cos series with $L=\pi$

$\Rightarrow \mu = \left(\frac{m\pi}{L}\right)^2 = m^2$

Eigenfunction $\cos(m\theta)$

ODE BVP $\begin{cases} r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - m^2) f = 0 \\ f'(a) = 0 \\ |f(0)| < \infty \end{cases}$

Rayleigh quotient:

$p = r, \quad \sigma = r, \quad q = -\frac{m^2}{r}$

$$\lambda = \frac{-r f \frac{df}{dr}\Big|_0^a + \int_0^a \left(r \left(\frac{df}{dr}\right)^2 + \frac{m^2}{r} f^2 \right) dr}{\int_0^a f^2 r \, dr} \geq 0$$

To have $\lambda = 0$ we need ~~$f = 0 \Rightarrow$ excluded~~ $m = 0$ and $f = \text{const} = 1$

~~$\Rightarrow \lambda > 0$~~

If $\lambda > 0$

$$f = c_1 J_m(\sqrt{\lambda}\, r) + c_2 Y_m(\sqrt{\lambda}\, r)$$

$c_2 = 0$ by singularity condition $|f(0)| < \infty$

$\Rightarrow$ ~~eigenfunct~~

$$f(a) = c_1 J_m(\sqrt{\lambda}\, a) = 0$$

$$\Rightarrow \lambda = \left(\frac{z_{mn}}{a}\right)^2, \quad z_{mn} - \text{ nth zero of } J_m$$

Define eigenfunction as

$$\phi_{mn}(r) = \begin{cases} 1, & m = 0, n = 1 \\ J_m(\sqrt{\lambda_{mn}}\, r), & m, n = 1, 2, \ldots \end{cases}$$

Superposition:

$$u(r, \theta, t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} A_{mn} \phi_{mn}(r) \cos(m\theta) e^{-\lambda_{mn} k t}$$

IC: $u(r,\theta,0) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} A_{mn} \Phi_{mn}(r) \cos(m\theta)$

By 2D orthogonality.

$$A_{mn} = \frac{\int_0^{\pi} \int_0^{a} f(r,\theta)\, \Phi_{mn}(r) \cos(m\theta)\, r\, dr\, d\theta}{\int_0^{\pi} \int_0^{a} \Phi_{mn}^2(r) \cos^2(m\theta)\, r\, dr\, d\theta}$$

Ans $t \to \infty$ ~~∞~~ $\Rightarrow e^{-\lambda_{mn} k t} \to 0$ except $m=0, n=1$ for $\lambda_{mn} = 0$.

$\Rightarrow u(r,\theta,t) \Rightarrow A_{01}$, where

$$A_{01} = \frac{\int_0^{\pi} \int_0^{a} f(r,\theta)\, r\, dr\, d\theta}{\frac{1}{2}\pi a^2}$$

the equilibrium ~~at~~ $t \to \infty$ is nonzero

~~becau~~ because of insulated BC,

i.e. thermal energy is conserved.

7.7.10

$$\begin{cases} \frac{\partial u}{\partial t} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \\ u(a,t) = 0 \\ u(r,0) = f(r) \end{cases}$$

$$u = f(r)\,h(t)$$

$$\frac{h(t)}{dt} = -k\lambda h \Rightarrow h = c e^{-k\lambda t}$$

$$r^2\frac{d^2f}{dr^2} + r\frac{df}{dr} + \lambda r^2 f = 0$$

$$\lim_{r\to 0} |f(r)| < \infty \Rightarrow f(r) = c_1 J_0(\sqrt{\lambda}\, r)$$

$$f(a) = c_1 J_0(\sqrt{\lambda}\, a) = 0$$

$$\Rightarrow \lambda = \left(\frac{z_n}{a}\right)^2, \quad n = 1, 2, \ldots$$

z_n - nth zero of J_0.

Superposition

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\sqrt{\lambda_n}\, r)\, e^{-k\lambda t}}$$

I.C. $$u(r,0) = \sum_{n=1}^{\infty} A_n J_0(\sqrt{\lambda_n}\, r)$$

$\Rightarrow$ Fourier - Bessel series orthogonality

$$\boxed{A_n = \frac{\int_0^a f(r) J_0(\sqrt{\lambda_n}\, r)\, r\,dr}{\int_0^a J_0^2(\sqrt{\lambda_n}\, r)\, r\,dr}}$$

As $t \to \infty \Rightarrow u \to 0$ since $\lambda_1 > 0$.

7.7.12

(a) $$x^2 \frac{d^2y}{dx^2} + (x-6)y = 0$$

for $x \to 0$

$$x^2 y'' - 6y = 0$$

Substituting $y = x^p$

$$\Rightarrow p(p-1) - 6 = 0$$

$$\Rightarrow p = -2, 3$$

$$\Rightarrow y \approx c_1\left(\frac{1}{x^2} + \dots\right) + c_2\left(x^3 + \dots\right)$$

(c) $$x^2 y'' + (x + x^2) y' + 4y = 0$$

$$x \to 0 \Rightarrow x^2 y'' + x y' + 4y = 0$$

Substituting $y = x^p$

$$\Rightarrow p(p-1) + p + 4 = 0$$

$$\Rightarrow p = \pm 2i$$

Using $x^{\pm 2i} = e^{\pm 2i \ln x} = \cos(2\ln x) \pm i\sin(2\ln x)$

$$\Rightarrow y = c_1\left(\cos(2\ln x) + \dots\right) + c_2\left(i\sin(2\ln x) + \dots\right)$$