math 312 HW 10

7.8.1

$$\frac{d}{dr}\left(r\frac{df}{dr}\right)+\left(\lambda r-\frac{m^2}{r}\right)f=0$$

$$f(1)=f(2)=0, \quad m=0,1,2\ldots$$

(a) SLDE with $x=r$, $\sigma=r$, $p=r$,
$\phi=f$

$$q=-\frac{m^2}{r}$$

Rayleigh quotient

$$\lambda=\frac{-\overbrace{p\phi\frac{d\phi}{dr}\Big|_1^2}^{=0}+\int_1^2\left[r\left(\frac{d\phi}{dr}\right)^2-q\phi^2\right]dr}{\int_1^2\sigma\phi^2\,dr}$$

$$\int_1^2 r\left(\frac{d\phi}{dr}\right)^2=\frac{\int_1^2\left(r\left(\frac{df}{dr}\right)^2+\frac{m^2}{r}f^2\right)dr}{\int_1^2 r f^2\,dr}\geq 0$$

If $\lambda=0 \Rightarrow \frac{df}{dr}=0 \Rightarrow f=\text{const}=0$ (by BC)
$\Rightarrow$ only trivial sol $\Rightarrow \lambda>0$.

(b) Because $\lambda > 0$

$$\Rightarrow \quad f = c_1 J_m(\sqrt{\lambda}\, r) + c_2 Y_m(\sqrt{\lambda}\, r)$$

$$\text{BCs:} \quad \begin{cases} f(1) = c_1 J_m(\sqrt{\lambda}) + c_2 Y_m(\sqrt{\lambda}) = 0 \\ f(2) = c_1 J_m(2\sqrt{\lambda}) + c_2 Y_m(2\sqrt{\lambda}) = 0 \end{cases}$$

$\Rightarrow$ by elimination of $c_1 (c_2)$

$$J_m(\sqrt{\lambda})\, Y_m(2\sqrt{\lambda}) - J_m(2\sqrt{\lambda})\, Y_m(\sqrt{\lambda}) = 0$$

– eq. for eigenvalues.

(c) use Rayleigh quotient:

$$\lambda = \frac{\int_1^2 r \left(\frac{d f}{dr}\right)^2 dr + m^2 \int_1^2 \frac{f^2}{r}\, dr}{\int_1^2 f^2 r\, dr}$$ – smallest for $\underline{m = 0}$

(d) Using the Rayleigh minimization principle (5.6.5) with $p = r$, $q = -\frac{m^2}{r}$, $\sigma = r$, $x = r$:

$$\lambda_1 = \min \frac{\int_1^2 \left[r\left(\frac{du}{dr}\right)^2 + \frac{m^2 u^2}{r} \right] dr}{\int_1^2 u^2 r\, dr}$$

(we take $m = 0$ from (c))

$$= \min \frac{\int_1^2 r\left(\frac{du}{dr}\right)^2 dr}{\int_1^2 u^2 r\, dr}$$

Use (5.7.13):

$$\min \frac{\int_1^2 1 \cdot \left(\frac{du}{dr}\right)^2 dr}{\int_1^2 2 \cdot u^2 dr} \le \lambda_1 \le \frac{\int_1^2 2\left(\frac{du}{dr}\right)^2 dr}{\int_1^2 u^2 dr}$$

But $\min \dfrac{\int_1^2 \left(\frac{du}{dr}\right)^2 dr}{\int_1^2 u^2 dr} = \left(\dfrac{\pi}{L}\right)^2$, where $L = 2 - 1$, (from Rayleigh quotient for $\begin{cases} \frac{d^2u}{dx^2} = -\hat{\lambda} \\ u(1) = u(2) = 0 \end{cases}$)

$$\Rightarrow \boxed{\frac{1}{2}\pi^2 \le \lambda_1 \le 2\pi^2}$$

7.8.2

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$


$\theta = 0$

$$u(r,0,t) = u(a,\theta,t) = u(r,\tfrac{\pi}{2},t) = 0$$

$$u(r,\theta,0) = G(r,\theta)$$

(a) $u(r,\theta,t) = \varphi(r,\theta)\, h(t)$

$$\Rightarrow \frac{dh}{dt} = -\lambda k h \Rightarrow h(t) = c\, e^{-\lambda k t}$$

$$\nabla^2\varphi + \lambda\varphi = 0, \quad \varphi = f(r)\, g(\theta)$$

$$\Rightarrow \begin{cases} \dfrac{d^2 g}{d\theta^2} = -\mu g \\ g(0) = g(\tfrac{\pi}{2}) = 0 \end{cases} \Rightarrow \mu = \left(\frac{m\pi}{L}\right)^2 = \left(\frac{m\pi}{\frac{\pi}{2}}\right)^2 = 4m^2,$$

$m = 1, 2, \ldots$

Eigenfunctions $\sin(2m\theta)$

$$\Rightarrow \frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{\mu}{r}\right) f = 0, \text{ where } \mu = 4m^2$$

(b) If $\mu \geq 0$ then

$$\lambda = \frac{\int_0^a \left[r\left(\frac{df}{dr}\right)^2 + \frac{m^2}{r} f^2\right] dr}{\int_0^a r f^2\, dr} > 0$$

(see 7.8.1)

(c) Because $|f(0)| < \infty$

$\Rightarrow \quad f(r) = C_1 J_{2m}(\sqrt{\lambda}\, r)$

BC: $\quad J_{2m}(\sqrt{\lambda}\, a) = 0$

$\Rightarrow$ smallest eigenvalue corresponds to the first zero of $J_{2m}(\sqrt{\lambda}\, r)$

$\Rightarrow$ no zero inside $0 < r < a$.

(d) Assume z_{mn} – nth zero of J_{2m}

$\Rightarrow \lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$

Superposition:

$$u(r,\theta,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{mn} J_{2m}(\sqrt{\lambda_{mn}}\, r) \sin(2m\theta)$$

$$\times\ e^{-\lambda_{mn} k t}.$$

IC: $\quad u(r,\theta,0) = G(r,\theta) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{mn} J_{2m}(\sqrt{\lambda_{mn}}\, r) \sin(2m\theta)$

2D Orthogonality:

$$C_{mn} = \frac{\int_0^{\pi/2}\int_0^{a} f(r,\theta)\, J_{2m}(\sqrt{\lambda_{mn}}\, r)\sin(2m\theta)\, d\theta\, r\, dr}{\int_0^{\pi/2}\int_0^{a} J_{2m}^2(\sqrt{\lambda_{mn}}\, r)\sin^2(2m\theta)\, d\theta\, r\, dr}$$

7.9.1 b

$\nabla^2 u = 0$ in 3D inside a circular cylinder

$$u(r,\theta,0) = \alpha(r)\sin(7\theta),\quad u(r,\theta,H) = 0,\quad u(a,\theta,z) = 0$$

$$u(r,\theta,z) = f(r)\,g(\theta)\,h(z)$$

Similar to section 7.9.3:

$$\frac{1}{rf}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{gr^2}\frac{d^2g}{d\theta^2} = -\frac{1}{h}\frac{d^2h}{dz^2} = -\lambda$$

We have nonuniform BC in $z \Rightarrow$ first solve in r and θ.

$$r^2 \Big| \quad \frac{1}{rf}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{gr^2}\frac{d^2g}{d\theta^2} = -\lambda$$

$$\Rightarrow -\frac{1}{g}\frac{d^2g}{d\theta^2} = \mu \Rightarrow \begin{cases} \dfrac{d^2g}{d\theta^2} + \mu g = 0 \\ g(0) = g(2\pi) \end{cases} \Rightarrow \mu_m = m^2,\ m = 0, 1, \ldots$$

$$g_m = \cos(m\theta),\ \sin(m\theta)$$

$$r^2\frac{d^2f}{dr^2} + r\frac{df}{dr} + (\lambda r^2 - m^2)f = 0$$

$$\Rightarrow f = c_1 J_m(\sqrt{\lambda}r) + \underbrace{c_2}_{\substack{= \\ 0}} Y_m(\sqrt{\lambda}r)$$

by $|f(0)| < \infty$

$$\Rightarrow J(\sqrt{\lambda_{mn}}\, a) = 0$$

$$\frac{d^2 h}{dz^2} - \lambda h = 0 \Rightarrow h(z) = c_1 \cosh\left[\sqrt{\lambda}(z-H)\right] + c_2 \sinh\left(\sqrt{\lambda}(z-H)\right)$$

shift because $h(H)=0$

$$h(H) = 0 \qquad \Rightarrow c_2 = 0$$

Superposition:

$$u(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left[\cos(m\theta) + B_{mn}\sin(m\theta)\right] \times J_m(\sqrt{\lambda_{mn}}\, r)\sinh\left(\sqrt{\lambda_{mn}}(z-H)\right)$$

at $z=0$:

$$u(r,\theta,0) = \alpha(r)\sin(7\theta) = \sum_{n=1}^{\infty} B_{7n} J_7(\sqrt{\lambda_{7n}}\, r)\sin(7\theta)\sinh\left(\sqrt{\lambda_{7n}}(z-H)\right)$$

} because of orthogonality in θ

$\Rightarrow$ orthogonality in r:

$$-\sinh(\sqrt{\lambda_{7n}}H)\, B_{7n} = \frac{\int_0^a \alpha(r) J_7(\sqrt{\lambda_{7n}}\, r)\, r\, dr}{\int_0^a J_7^2(\sqrt{\lambda_{7n}}\, r)\, r\, dr}, \quad J_7(\sqrt{\lambda_{7n}}\, a) = 0$$

$$u(r,\theta,z) = \sum_{n=1}^{\infty} B_{7n} J_7(\sqrt{\lambda_{7n}}\, r)\sin(7\theta) \times \sinh\left(\sqrt{\lambda_{7n}}(z-H)\right)$$

7.9.2a $\quad \nabla^2 u = 0$ in a semicircular cylinder.

$$u(r,\theta,0)=0, \quad u(r,\theta,H)=\alpha(r,\theta),$$

$$u(r,0,z)=0, \quad u(r,\pi,z)=0, \quad u(a,\theta,z)=0$$

Separation of variables as above.

$$\frac{d^2 g}{d\theta^2} + \mu g = 0 \Rightarrow g(\theta) = c_1 \cos(\sqrt{\mu}\,\theta) + c_2 \sin(\sqrt{\mu}\,\theta)$$

$$g(0) = 0 = c_1 \Rightarrow g(\pi) = c_2 \sin\sqrt{\mu}\pi = 0$$

$$\Rightarrow \sqrt{\mu_m}\,\pi = \pi m, \quad m = 1, 2, \ldots$$

$$\boxed{\begin{array}{l} \mu_m = m^2 \\ g_m(\theta) = \sin(m\theta) \end{array}}$$

$$r^2 \frac{d^2 f}{dr^2} + r\frac{df}{dr} + (\lambda r^2 - m^2) = 0$$

$$\Rightarrow f_{mn}(r) = B_1 J_m(\sqrt{\lambda_{mn}}\, r), \quad (Y_m \text{ excluded by } |f(0)| < \infty)$$

$$f_{mn}(a) = J_m(\sqrt{\lambda_{mn}}\, a) = 0, \quad n = 1, 2, \ldots$$

$$\frac{d^2 h}{dz^2} - \lambda_{mn} h = 0 \Rightarrow h(z) = \sinh(\sqrt{\lambda_{mn}}\, z)$$

for $h(0)=0$

Superposition :

$$u(r,\theta,z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}}\, r) \sin(m\theta) \times \sinh(\sqrt{\lambda_{mn}}\, z) \quad , \quad J_m(\sqrt{\lambda_{mn}}\, a) = 0$$

$u(r,\theta,H) = \alpha(r,\theta) \Rightarrow$ by orthogonality

$$A_{mn} = \frac{\int_0^a \int_0^\pi \alpha(r,\theta) J_m(\sqrt{\lambda_{mn}}\, r) \sin(m\theta)\, r\, dr\, d\theta}{\left(\int_0^a \int_0^\pi (J_m(\sqrt{\lambda_{mn}}\, r))^2 \sin^2(m\theta)\, r\, dr\, d\theta \right) \sinh(\sqrt{\lambda_{mn}}\, H)}$$

7.9.3.a $\quad \frac{\partial u}{\partial t} = k \nabla^2 u \quad , \quad 0 < r < a \quad , \quad 0 < \theta < \frac{\pi}{2} \quad ,$

$0 < z < H$

I C. $u(r,\theta,z,0) = f(r,\theta,z)$

B C $\quad u(r,\theta,0) = u(r,\theta,H) = u(r,0,z) = u(r,\frac{\pi}{2},z)$

$= u(a,\theta,z) = 0$

$$u(r,\theta,z,t) = f(r)\, g(\theta)\, q(z)\, h(t)$$

$$\frac{1}{rf}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{r^2 g}\frac{d^2 g}{d\theta^2} + \frac{1}{q}\frac{d^2 q}{dz^2} = \frac{1}{kh}\frac{dh}{dt} = -\lambda$$

$\frac{1}{r^2 g}\frac{d^2 g}{d\theta^2} = -\mu \qquad \frac{1}{q}\frac{d^2 q}{dz^2} = -\nu \Rightarrow h(t) = c e^{-\lambda k t}$

$$\begin{cases} \dfrac{d^2 q}{dz^2} = -\nu q \\ q(0) = q(H) = 0 \end{cases} \Rightarrow \nu_l = \left(\frac{l\pi}{H}\right)^2, \quad q_l = \sin\frac{l\pi z}{H}, \; l = 1, 2, \ldots$$

$$\begin{cases} \dfrac{d^2 g}{d\theta^2} + \mu g = 0 \\ g(0) = 0 \\ g\left(\frac{\pi}{2}\right) = 0 \end{cases} \Rightarrow \left(L = \frac{\pi}{2}\right) \quad \mu_m = \left(\frac{m\pi}{\frac{\pi}{2}}\right)^2 = (2m)^2, \quad g_m = \sin(2m\theta), \; m = 1, 2, \ldots$$

$$\frac{1}{rf}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{r^2}(-\mu_m) = -\lambda + \nu = -\lambda + \left(\frac{l\pi}{H}\right)^2$$

$$\Rightarrow r^2\frac{d^2 f}{dr^2} + r\frac{df}{dr} + \left[\lambda - \left(\frac{l\pi}{H}\right)^2\right] r^2 f - \mu_m f = 0$$

Because $|f(0)| < \infty \Rightarrow f_{nml} = J_{2m}\left(\sqrt{\hat{\lambda}_{nm}}\, r\right)$,

where $\hat{\lambda} \equiv \lambda - \left(\frac{l\pi}{H}\right)^2$

and $J_{2m}\left(\sqrt{\hat{\lambda}_{nm}}\, a\right) = 0, \; n = 1, 2, \ldots$

Superposition:

$$u(r, \theta, z, t) = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\sum_{n=1}^{\infty} A_{nml} J_{2m}\left(\sqrt{\hat{\lambda}_{nm}}\, r\right)\sin(2m\theta) \times \sin\left(\frac{\pi l z}{H}\right) e^{-\kappa\left[\hat{\lambda}_{nm} + \left(\frac{\pi l}{H}\right)^2\right] t}.$$

IC:

$$u(r,\theta,z,0) = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\sum_{n=1}^{\infty} A_{nml} J_{2m}\left(\sqrt{\tilde{\lambda}_{nm}}\, r\right) \sin(2m\theta) \sin\left(\frac{\pi l z}{H}\right)$$

orthogonality

$$\Rightarrow A_{nml} = \frac{\int_0^H \int_0^a \int_0^{\pi/2} f(r,\theta,z) J_{2m}\left(\sqrt{\tilde{\lambda}_{nm}}\, r\right) \sin(2m\theta) \sin\left(\frac{\pi l z}{H}\right) dz\, r\, dr\, d\theta}{\int_0^H \int_0^a \int_0^{\frac{\pi}{2}} \left[J_{2m}\left(\sqrt{\tilde{\lambda}_{nm}}\, r\right) \sin(2m\theta) \sin\left(\frac{\pi l z}{H}\right)\right]^2 dz\, r\, dr\, d\theta}$$

$A\ t \to 0 \Rightarrow u \to 0$ because $\tilde{\lambda}_{nm} + \left(\frac{\pi l}{H}\right)^2 > 0$ $\forall n, m, l$.

(because BC are zeros $\Rightarrow$ only equilibrium sol. is zero)

7.9.4 a

$\frac{\partial u}{\partial t} = k \nabla^2 u$ inside a cylinder of radius a and height H.

BC $u(r,\theta,0,t)=0$, $u(r,\theta,H,t)=0$

$u(a,\theta,z,t)=0$

IC $u(r,\theta,z,0)=f(r,z)$ - θ independent

By separation of variables in time:

$\nabla^2 \phi + \lambda \phi = 0$ and $\frac{dh}{dt} = -\lambda k h \Rightarrow h = ce^{-\lambda k t}$.

Separation of variables for ϕ:

$\phi = g(r)\,Q(z)$ - because there is no dependence on θ!

$$\Rightarrow \frac{1}{rg}\frac{d}{dr}\left(r\frac{dg}{dr}\right) + \underbrace{\frac{1}{Q}\frac{d^2Q}{dz^2}}_{= -\nu} = -\lambda$$

$$\begin{cases} \frac{d^2Q}{dz^2} = -\nu \\ Q(0) = Q(H) = 0 \end{cases} \Rightarrow \begin{matrix} \nu_\ell = \left(\frac{\ell\pi}{H}\right)^2 \\ Q = \sin\left(\frac{\ell\pi z}{H}\right), \ \ell = 1,2,\dots \end{matrix}$$

$$r^2 \frac{d^2 g}{dr^2} + r \frac{dg}{dr} + (\lambda - \nu_\ell) r^2 g = 0$$

Bessel DE of order 0.

$|g(0)| < \infty$

$$\Rightarrow g = c_1 J_0(\sqrt{\lambda - \nu_\ell}\, r)$$

$$g(a) = 0 \Rightarrow g_n = J_0(\sqrt{\tilde{\lambda}_n}\, a) = 0$$

$$\tilde{\lambda} \equiv \lambda - \nu_\ell = \lambda - \left(\frac{\ell\pi}{H}\right)^2$$

By superposition:

$$u(r,z,t) = \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} A_{n\ell} J_0(\sqrt{\tilde{\lambda}_n}\, r) \sin\left(\frac{\ell\pi z}{H}\right) e^{-\lambda k t},$$

where $\lambda = \tilde{\lambda}_n + \left(\frac{\ell\pi}{H}\right)^2$, $J_0(\sqrt{\tilde{\lambda}_n}\, a) = 0$

$$\text{IC} \Rightarrow A_{n\ell} = \frac{\int_0^H \int_0^a J_0(\sqrt{\tilde{\lambda}}\, r) \sin\left(\frac{\ell\pi z}{H}\right) r\,dr\,dz}{\int_0^H \int_0^a \left[J_0(\sqrt{\tilde{\lambda}_n}\, r) \sin\left(\frac{\ell\pi z}{H}\right)\right]^2 r\,dr\,dz}$$