HW 05 Solutions

4.4.1 Vibrating string of density ρ_0, tension T_0

(a) Fixed ends: $u(0,t)=u(L,t)=0$

$$u_{tt} = c^2 u_{xx}, \quad c^2 = \frac{T_0}{\rho_0}$$

Natural frequencies are $c\sqrt{\lambda}$ and

$\lambda = \left(\frac{n\pi}{L}\right)^2$ according to section 4.4.

$\Rightarrow$ natural frequencies $= \frac{n\pi c}{L}$, $n=1,2,\ldots$

(b) $u(0,t) = \frac{du(H)}{dx} = 0$

$u = \phi(x)\, h(t)$

ODE BVP
$$\begin{cases} \dfrac{d^2\phi}{dx^2} = -\lambda\phi \\ \phi(0)=0 \\ \dfrac{d\phi(H)}{dx} = 0 \end{cases}$$

$\Rightarrow \phi = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

$\phi(0) = c_1 = 0$

$\Rightarrow \phi(x) = c_2\sin(\sqrt{\lambda}x)$

$$\frac{d\phi}{dx} = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$$\frac{d\phi(H)}{dx} = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}H) = 0$$

$\lambda<0$, $\lambda=0$ – only trivial sol

$\Rightarrow H\sqrt{\lambda} = -\frac{\pi}{2} + \pi n$

$$\Rightarrow \lambda = \frac{\pi^2(n-\frac{1}{2})^2}{H^2}, \quad n=1,2,\ldots$$

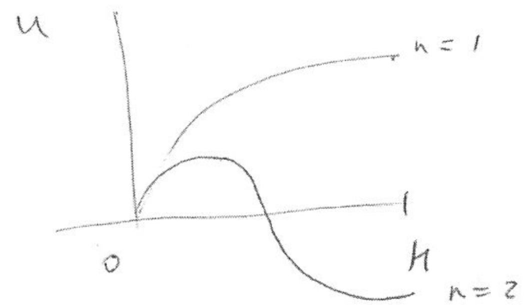
$$h'' = -\lambda c^2 h$$

$$\Rightarrow h = c_1 \cos(c\sqrt{\lambda} t) + c_2 \sin(c\sqrt{\lambda} t)$$

$c\sqrt{\lambda}$ – natural frequencies

$$\Rightarrow c\sqrt{\lambda} = \frac{(n-\frac{1}{2})\pi c}{H}, \quad n = 1, 2, \ldots$$

Eigenfunctions $\sin\left(\frac{(n-\frac{1}{2})\pi x}{H}\right)$



(c) Odd harmonics for (a):

$$u = \phi(x)\, h(t)$$

$H = \frac{L}{2}$

$$\begin{cases} \dfrac{d^2 h}{dt^2} = -\lambda h \\[2ex] \dfrac{d^2\phi}{dx^2} = -\lambda \phi \\ \phi(0) = \phi(L) = 0 \end{cases}$$

$\lambda = \left(\frac{n\pi}{L}\right)^2$. For n odd: $n = 1, 3, 5, \ldots$

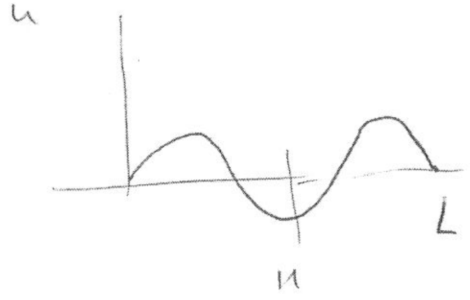
$$\lambda = \left(\frac{(2m-1)\pi}{L}\right)^2 = \frac{(m-\frac{1}{2})^2\pi^2}{(L/2)^2} = \frac{(m-\frac{1}{2})^2\pi^2}{H^2},$$

$m = 1, 2, \ldots$

$\Rightarrow$ frequences of (a) = (b) for $n = 2m-1$, $m = 1, 2, \ldots$

Symmetry: for n odd modes we have $n-1$, i.e. even number of zeros symmetric relative to $x = H$.

$\Rightarrow \frac{du(H)}{dx} = 0$.



4.4.3 $\quad \rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$.

(a) β is positive because damping slows down motion. E.g. for $\frac{\partial u}{\partial x} = 0$

$$\Rightarrow \rho_0 \frac{d^2 u}{dt^2} = -\beta \frac{du}{dt}$$

$$\Rightarrow \frac{du}{dt} = \frac{du}{dt}(0)\, e^{-\frac{\beta}{\rho_0} t}$$ - decays for $\beta > 0$ because $\rho_0 > 0$.

(b) BC $\quad u(0,t) = u(L,t) = 0$

IC $\quad u(x,0) = f(x)$ and $\frac{\partial u}{\partial x}(x,0) = g(x)$

Also assume $\beta^2 < 4\pi^2 \frac{\rho_0 T_0}{L^2}$

Separation of variables:

$$u = \varphi(x)\, h(t)$$

$$\Rightarrow \frac{\rho_0 \varphi \frac{d^2 h}{dt^2} = T_0 h \frac{d^2\varphi}{dx^2} - \beta \varphi \frac{dh}{dt}}{T_0 \varphi h}$$

$$\Rightarrow \frac{\rho_0}{T_0} \frac{\frac{d^2 h}{dt^2}}{h} + \frac{\beta}{T_0} \frac{\frac{dh}{dt}}{h} = \frac{\frac{d^2\varphi}{dx^2}}{\varphi} = -\lambda$$

ODE BVP: $\begin{cases} \frac{d^2\varphi}{dx^2} = -\lambda \varphi \\ \varphi(0) = \varphi(L) = 0 \end{cases}$ $\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$, $\varphi = \sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, \dots$

Equation for h:

$\rho_0 h'' + \beta h' + T_0 \lambda h = 0$ — homogeneous linear ODE with constant coefficients

$\Rightarrow$ solve in the form $h = e^{rt}$

$\Rightarrow$ characteristic eq:

$$\rho_0 r^2 + \beta r + \left(\frac{n\pi}{L}\right)^2 T_0 = 0$$

$$\Rightarrow r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 T_0 \left(\frac{n\pi}{L}\right)^2}}{2\rho_0}$$

Since $\beta^2 < 4\rho_0 T_0 \left(\frac{\pi}{L}\right)^2 \Rightarrow$ discriminant $< 0 \ \forall n$

$\Rightarrow$ complex eigenvalue!

$$r_{1,2} = -\frac{\beta}{2\rho_0} \pm i\sqrt{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4\rho_0^2}}$$

General sol:

$$h = c_1 e^{-\frac{\beta}{2\rho_0}t} \cos\left[\sqrt{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4\rho_0^2}}\, t\right]$$

$$+ c_2 e^{-\frac{\beta}{2\rho_0}t} \sin\left[\sqrt{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4\rho_0^2}}\, t\right]$$

Here we also see explanation for (a) that for $\beta < 0$ we would have exp growing solution which is unphysical for damping.

Superposition:

$$u = e^{-\frac{\beta}{2\rho_0}t} \sum_{n=1}^{\infty} \left[a_n \cos(w_n t) + b_n \sin(w_n t)\right] \sin\frac{n\pi x}{L},$$

where $w_n = \sqrt{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4\rho_0^2}}$.

IC: $u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$

$$\Rightarrow \boxed{a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = -\frac{\beta}{2\rho_0} e^{-\frac{\beta t}{2\rho_0}} \sum_{n=1}^{\infty}\left[a_n \cos(w_n t)\right.$$

$$\left.+ b_n \sin(w_n t)\right] \times \sin\left(\frac{n\pi x}{L}\right)\Big|_{t=0} + e^{-\frac{\beta t}{2\rho_0}} \sum_{n=1}^{\infty} w_n\left[-a_n \sin(w_n t)\right.$$

$$\left.+ b_n \cos(w_n t)\right] \sin\left(\frac{n\pi x}{L}\right)\Big|_{t=0}$$

$$= -\frac{\beta}{2\rho_0}\underbrace{\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)}_{f(x)} + \sum_{n=1}^{\infty} b_n w_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \boxed{b_n w_n = \frac{\beta a_n}{2\rho_0} + \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

already given above

5.3.3

$$H(x) \Big| \frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)]\phi = 0$$

We need to find $H(x)$ such that the resulting eq is in SL form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\phi = 0$$

$$H(x)\frac{d^2\phi}{dx^2} + H(x)\alpha(x)\frac{d\phi}{dx} = \frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right]$$

$$= p'\frac{d\phi}{dx} + p\frac{d^2\phi}{dx^2}$$

$$\Rightarrow \left.\begin{array}{l} p'(x) = H(x)\alpha(x) \\ p(x) = H(x) \end{array}\right\} \Rightarrow H' = H\alpha$$

$$\frac{dH}{H} = \alpha(x)\,dx$$

$$\Rightarrow H(x) = c_1 e^{\int^x \alpha(t)dt}$$

Set e.g. $c_1 = 1 \Rightarrow \boxed{H = e^{\int^x \alpha(t)dt}}$

$$\Rightarrow p(x) = H(x),\quad q(x) = \gamma H,\quad \sigma = \beta H.$$

5.3.5 $\varphi'' + \lambda \varphi = 0$, $\frac{d\varphi}{dx}(0) = \frac{d\varphi}{dx}(L) = 0$ (8)

(a)

$$\Rightarrow \quad \varphi = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$$

$$\frac{d\varphi}{dx} = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} x) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x)$$

$\lambda > 0$

$$\frac{d\varphi(0)}{dx} = 0 = c_2 \sqrt{\lambda} \Rightarrow c_2 = 0$$

$$\frac{d\varphi}{dx}(L) = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} L) = 0$$

$\lambda = 0$ $\quad \varphi = \text{const}$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \quad \varphi_n = \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, \ldots$$

(a) ∞ of eigenvalues.

(b) $\cos\left(\frac{n\pi x}{L}\right)$ has $n-1$ zeros at $0 < x < L$

(c) Fourier cosine series ~~has~~ use orthogonality of cosines. Completness - from thm. on Fourier Series.

not assigned

(d) Rayleigh quotient

$$\lambda = \frac{\overbrace{-\varphi \frac{d\varphi}{dx}\Big|_0^L}^{=0} + \int_0^L \left(\frac{d\varphi}{dx}\right)^2 dx}{\int_0^L \varphi^2 \, dx} \geq 0 \Rightarrow \lambda < 0 \text{ is not eigenvalue}$$

$\lambda = 0$ is eigenvalue for $\varphi = \text{const}$.

5.3.9.c

$$x^2 \frac{d^2\Phi}{dx^2} + x\frac{d\Phi}{dx} + \lambda\Phi = 0$$

$$\Phi(1) = 0, \quad \phi(b) = 0$$

Since eq. is equidimensional $\Rightarrow \Phi = x^r$

$\Rightarrow \quad r(r-1) + r + \lambda = 0 \Rightarrow r^2 = -\lambda$

$\underline{\lambda > 0}$ $\quad r = \pm i\sqrt{\lambda} \Rightarrow x^r = x^{\pm i\sqrt{\lambda}} = e^{\pm i\sqrt{\lambda}\ln x}$

$\Rightarrow$ general sol $\Phi = c_1 \cos(\sqrt{\lambda}\ln x) + c_2 \sin(\sqrt{\lambda}\ln x)$

$\underline{BC}$ $\quad \Phi(1) = 0 = c_1$

$\Phi(b) = c_2 \sin(\sqrt{\lambda}\ln b) = 0 \Rightarrow \sqrt{\lambda}\ln b = n\pi, \quad n = 1, 2, \ldots$

$\Rightarrow \lambda = \left(\frac{n\pi}{\ln b}\right)^2$

Eigenfunction $\sin\left(\frac{n\pi}{\ln b}\ln x\right)$

$\underline{\lambda = 0}$ $\Rightarrow \quad x^2\frac{d^2\Phi}{dx^2} + x\frac{d\Phi}{dx} = x\frac{d}{dx}\left(x\frac{d\Phi}{dx}\right) = 0$

$$\Rightarrow \frac{d}{dx}\left(x\frac{d\Phi}{dx}\right) = 0$$

$$x \frac{d\Phi}{dx} = c_1$$

$$d\Phi = c_1 \frac{dx}{x}$$

$$\Phi = c_1 \ln x + c_2$$

BC

$$\Phi(1) = 0 = c_2$$

$$\Phi(b) = c_1 \ln b = 0 \Rightarrow c_1 = 0 \Rightarrow \text{not eigenvalue}$$

$\underline{\lambda < 0}$ ~~not asked to analyse~~. $r = e^{\pm\sqrt{\lambda}\ln x}$

general sol

$$\Phi = c_1 e^{\sqrt{\lambda}\ln x} + c_2 e^{-\sqrt{\lambda}\ln x}$$

$$\Phi(1) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$\Phi(b) = c_1 \big(\underbrace{e^{\sqrt{\lambda}\ln b} - e^{-\sqrt{\lambda}\ln b}}_{\neq 0}\big) = 0$$

$\Rightarrow c_1 = 0$ – only trivial sol

So $\lambda_n = \left(\frac{n\pi}{\ln b}\right)^2 \to \infty$ as $n \to \infty$

Smallest λ: $\lambda_1 = \left(\frac{\pi}{\ln b}\right)^2$

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5.5.1 $p\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)\Big|_a^b=0 \Rightarrow$ self-adjoint

(c) $\frac{d\phi}{dx}(0)-h\phi(0)=0\ ,\ \frac{d\phi}{dx}(L)=0$

$\Rightarrow$ both for u and v $(a=0,\ b=L)$:

$$\frac{du(0)}{dx}-h\,u(0)=0=\cancel{\frac{d\phi}{dx}}\frac{du}{dx}(L)=0$$

$$\frac{dv(0)}{dx}-h\,v(0)=0=\frac{dv(L)}{dx}$$

$$\Rightarrow u\left(\frac{dv}{dx}-v\frac{du}{dx}\right)\Big|_0^L=u(L)\underbrace{\frac{dv(L)}{dx}}_{=0}-v(0)\underbrace{\frac{du(0)}{dx}}_{=0}$$

$$-u(0)\underbrace{\frac{dv(0)}{dx}}_{=h\,v(0)}+v(0)\underbrace{\frac{du(0)}{dx}}_{=h\,u(0)}=-h\,u(0)\,v(0)+h\,v(0)\,u(0)=0$$

(g) $\phi(L)+\alpha\phi(0)+\beta\frac{d\phi}{dx}(0)=0$

$\frac{d\phi}{dx}(L)+\gamma\phi(0)+\delta\frac{d\phi}{dx}(0)=0$

Solving for $u(L)$, $v(L)$; $\frac{du(L)}{dx}$, $\frac{dv(L)}{dx}$:

(2)

u:
$$\left.\begin{aligned} u(L) &= -\alpha\, u(0) - \beta \frac{du(0)}{dx} \\ \frac{du(L)}{dx} &= -\gamma\, u(0) - \delta \frac{du(0)}{dx} \\ v(L) &= -\alpha\, v(0) - \beta \frac{dv(0)}{dx} \\ \frac{dv(L)}{dx} &= -\gamma\, v(0) - \delta \frac{dv(0)}{dx} \end{aligned}\right\} \quad (*)$$
v:

$(a=0,\ b=L)$

$$\Rightarrow \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) \Big|_0^L$$

$$= u(L)\frac{dv(L)}{dx} - u(0)\frac{dv(0)}{dx} - v(L)\frac{du(L)}{dx} + v(0)\frac{du(0)}{dx}$$

$$= \{ \text{using } (*) \}$$

$$= \left(-\alpha\, u(0) - \beta\frac{du(0)}{dx}\right)\left(-\gamma\, v(0) - \delta\frac{dv(0)}{dx}\right) - u(0)\frac{dv(0)}{dx}$$
$$-\left(-\alpha\, v(0) - \beta\frac{dv(0)}{dx}\right)\left(-\gamma\, u(0) - \delta\frac{du(0)}{dx}\right) + v(0)\frac{du(0)}{dx}$$

$$= \alpha\gamma u(0)v(0) + \beta\gamma \frac{du(0)}{dx} v(0) + \alpha\delta u(0)\frac{dv(0)}{dx} + \beta\delta \frac{du(0)}{dx}\frac{dv(0)}{dx}$$

$$- u(0)\frac{dv(0)}{dx}$$

$$- \alpha\gamma v(0)u(0) - \beta\gamma \frac{dv(0)}{dx} u(0) - \alpha\delta v(0)\frac{du(0)}{dx} - \beta\delta \frac{dv(0)}{dx}\frac{du(0)}{dx}$$

$$+ v(0)\frac{du(0)}{dx}$$

$$= \frac{du(0)}{dx} v(0)\left[\beta\gamma - \alpha\delta + 1\right] + u(0)\frac{dv(0)}{dx}\left[\alpha\delta - 1 - \beta\gamma\right] = 0$$

$$\Rightarrow \boxed{\alpha\delta - \beta\gamma - 1 = 0}$$

5.5.5 $L = \frac{d^2}{dx^2} + 6\frac{d}{dx} + 9$

(a) $L(e^{rx}) = r^2 e^{rx} + r6e^{rx} + 9e^{rx}$

$= (r+3)^2 e^{rx}$

(b) $L(y) = 0$ using part (a): $r = -3$ double root

$\Rightarrow y = (c_1 + c_2 x)e^{-3x}$

②

5.3.3 $\quad \frac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad \frac{d\phi}{dx}(0) = 0$

(a) $\quad \frac{d\phi}{dx}(L) + h\,\phi(L) = 0,$

$h > 0$

$$\lambda = \frac{-p\phi\frac{d\phi}{dx}\Big|_0^L + \int_0^L \left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right]dx}{\int_0^L \phi^2 \sigma\, dx}$$

$$= \left\{ \begin{array}{c} p=1,\ \sigma=1,\ q=0 \\ \frac{d\phi}{dx}(L) = -h\phi(L) \end{array} \right\}$$

$$= \frac{h\,\phi(L)^2 + \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^L \phi^2\, dx} \geq 0 \quad \text{because } h > 0$$

Also in to have $= 0$ require $\phi(L) = 0$, $\frac{d\phi}{dx} = 0$

$\Rightarrow \phi(x) \equiv 0$ - only

trivial sol $\Rightarrow$ $\boxed{\lambda > 0}$

(b) By (a) $\lambda > 0$

$\Rightarrow \phi = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

$$\frac{d\Phi}{dx} = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$$\frac{d\Phi}{dx}(0) = 0 = c_2\sqrt{\lambda} \Rightarrow c_2 = 0$$

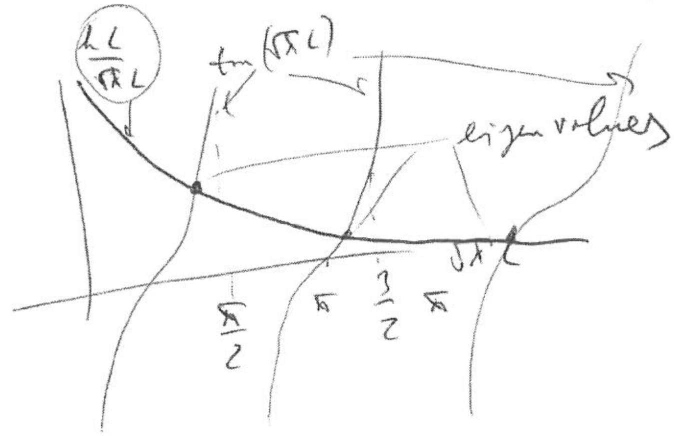
$$\Phi(x) = c_1\cos(\sqrt{\lambda}x)$$

BC at $x = L$: $\frac{d\Phi}{dx}(L) + h\,\Phi(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L)$

$$+ h\,c_1\cos(\sqrt{\lambda}L) = 0$$

$$\Rightarrow \tan(\sqrt{\lambda}L) = \frac{h}{\sqrt{\lambda}} = \frac{hL}{\sqrt{\lambda}L}$$

$\Rightarrow$



$\frac{hL}{\sqrt{\lambda}L}$ = hyperbola in $\sqrt{\lambda}L$

$$0 < \sqrt{\lambda_1}L < \frac{\pi}{2}$$

$$\pi < \sqrt{\lambda_2}L < \frac{3}{2}\pi$$

$$2\pi < \sqrt{\lambda_3}L < \frac{5}{2}\pi$$

$$\Rightarrow \boxed{(n-1)\pi < \sqrt{\lambda_n}\,L < \left(n-\frac{1}{2}\right)\pi}$$

$n = 1, 2, \ldots$

As $n \to \infty \Rightarrow \lambda_n \to \left(\frac{\pi}{2}(n-1)\right)^2$

Approach lower ~~band~~ bound

(4)

5.8.7

$$\frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\varphi(0) = 0, \quad \varphi(\pi) = 2\frac{d\varphi}{dx}(0)$$

(a) $$\int_0^\pi \left(u\frac{d^2v}{dx^2} - v\frac{d^2u}{dx^2} \right) dx$$

$$= u\frac{dv}{dx} - v\frac{du}{dx}\Big|_0^\pi - \underbrace{\int_0^\pi \left(\frac{du}{dx}\frac{dv}{dx} - \frac{dv}{dx}\frac{du}{dx} \right) dx}_{=0}$$

$$= u(\pi)\frac{dv(\pi)}{dx} - v(\pi)\frac{du(\pi)}{dx} = \underbrace{2\frac{du(0)}{dx}\cdot\frac{dv(\pi)}{dx} - 2\frac{dv(0)}{dx}\frac{du(\pi)}{dx}}_{\text{generally} \neq 0}$$

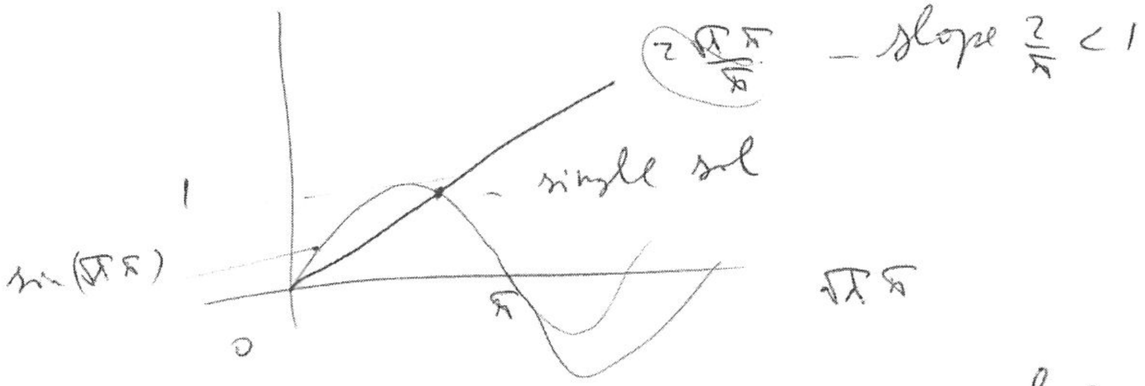
generally $\neq 0$

(for general values of $\frac{du(0)}{dx}$, $\frac{dv(0)}{dx}$, $\frac{dv(\pi)}{dx}$, $\frac{du(\pi)}{dx}$.

(b) If $\lambda > 0 \Rightarrow \varphi(0) = 0$ results in $\varphi = \sin(\sqrt{\lambda}x)$

$$\varphi(\pi) - 2\frac{\varphi(0)}{dx} = \sin(\sqrt{\lambda}\pi) - 2\sqrt{\lambda}\underbrace{\cos 0}_{=1} = \sin(\sqrt{\lambda}\pi) - 2\sqrt{\lambda} = 0$$

Graphical sol:



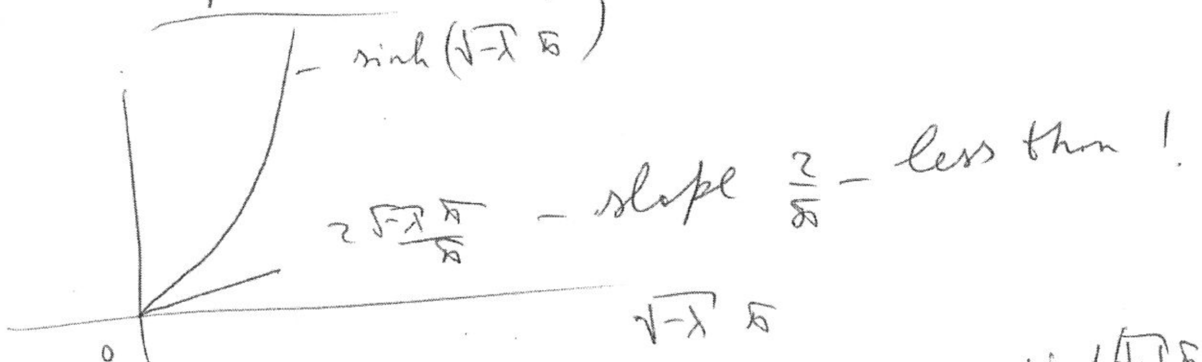
$\Rightarrow$ 1 positive eigenvalue < 1.

(c) If $\lambda < 0 \Rightarrow \phi(0) = 0$.

gives $\phi = \sinh(\sqrt{-\lambda}\, x)$

$$\phi(\pi) - 2\frac{\phi(0)}{dx} = \sinh(\sqrt{-\lambda}\,\pi) - 2\sqrt{-\lambda} = 0$$

Graphical sol:



$\Rightarrow$ no solution of $\sinh(\sqrt{-\lambda}\,\pi) = 2\sqrt{-\lambda}$ for $\lambda < 0$.

(d) $\lambda = 0 \Rightarrow \phi = c_1 x + c_2$

$\phi(0) = c_2 = 0$

$\Rightarrow \phi(x) = c_1 x$

$\phi(\pi) - 2\frac{d\phi(0)}{dx} = c_1 \pi - 2c_1 = 0$

$\Rightarrow c_1 = 0$ - only trivial sol

$\Rightarrow \lambda = 0$ is not eigenvalue.

(e) Since BC. are not in the form eq 5.3.2 of book, then this is not a regular Sturm-Liouville eigenvalue problem $\Rightarrow$ can have complex eigenvalues.

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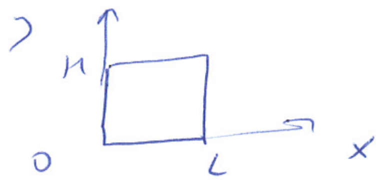
7.3.1



(a)

$$u_t = k(u_{xx}+u_{yy})$$

IC: $u(x,y,0) = \alpha(x,y)$

$$u = \phi(x,y)h(t) = f(x)g(y)h(t)$$

$$\frac{dh}{dt} = -\lambda k h$$

$$\nabla^2\phi + \lambda\phi = 0 \Rightarrow \frac{d^2f}{dx^2} = -\mu f$$

$$\frac{d^2g}{dy^2} + (\lambda-\mu)g = 0$$

(a) $u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,H,t) = 0$

$$\Rightarrow \begin{cases} \dfrac{d^2f}{dx^2} = -\mu f \\ f(0) = f(L) = 0 \end{cases} \Rightarrow \mu = \left(\frac{n\pi}{L}\right)^2, \; n=1,2,\ldots$$

Eigenfunctions:

$$f_n = \sin\left(\frac{n\pi x}{L}\right)$$

$$\begin{cases} \dfrac{d^2 g}{dy^2} = -(\lambda - \mu) g \\ g(0) = g(H) = 0 \end{cases} \Rightarrow \lambda_{nm} - \mu = \left(\frac{m\pi}{H}\right)^2$$

Eigenfunction:

$$g_{nm} = \sin\frac{m\pi y}{H}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

$$\varphi = \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}, \quad n, m = 1, 2, \ldots$$

Superposition:

$$\boxed{u = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} e^{-\lambda_{nm} k t}}$$

I.C. $$f = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L}$$

From orthogonality:

$$\boxed{a_{nm} = \frac{4}{LH} \int_0^H \int_0^L f \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right) dx\, dy}$$

As $t \to \infty \Rightarrow u \to 0$ (because all $\lambda > 0$)

the only equilibrium temperature with $u|_\Gamma = 0$.

(c) $$\frac{\partial u}{\partial x}(0,y,t) = \frac{\partial}{\partial x}(L,y,t) = 0$$

$$u(x,0,t) = 0 \ , \ u(x,H,t) = 0$$

$$u = \phi(x,y)\, h(t)$$

$$\frac{dh}{dt} = -\lambda k h$$

$$\nabla^2 \phi = -\lambda \phi$$

Separ. of variables: $\phi = f(x)\, g(y)$

$$\Rightarrow \begin{cases} \dfrac{d^2 f}{dx^2} = -\mu f \\ f'(0) = f'(L) = 0 \end{cases} \Rightarrow \mu = \left(\frac{n\pi}{L}\right)^2$$

Eigenfunction:

$$f = \cos\frac{n\pi x}{L} \ , \quad n = 0, 1, \ldots$$

$$\begin{cases} \dfrac{d^2 g}{dy^2} = -(\lambda - \mu) g \\ g(0) = g(H) = 0 \end{cases} \Rightarrow \lambda - \mu = \left(\frac{m\pi}{H}\right)^2, \ m = 1, 2, \ldots$$

$$\Rightarrow \lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

Eigenfunction:

$$\phi = \cos\frac{n\pi x}{L} \sin\frac{m\pi y}{H}, \quad n = 0, 1, \ldots \quad m = 1, 2, \ldots$$

Superposition:

$$u = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a_{nm} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) e^{-\lambda_{nm} k t}$$

$$\text{I C}: \quad f = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a_{nm} \cos\left(\frac{n\pi x}{L}\right) \sin\frac{m\pi y}{H}$$

By orthogonality:

$$a_{nm} = \frac{\int_0^H \int_0^L f \cos\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, dx\, dy}{\int_0^H \int_0^L \cos^2\left(\frac{n\pi x}{L}\right) \sin^2\left(\frac{m\pi y}{H}\right) dx\, dy}$$

here $\boxed{\int_0^H \int_0^L \cos^2\left(\frac{n\pi x}{L}\right) \sin^2\left(\frac{m\pi y}{H}\right) dx\, dy = \begin{cases} \frac{1}{2} LH, & n = 0 \\ \frac{1}{4} LH, & n \geq 1 \end{cases}}$

As $t \to \infty \Rightarrow u \to 0$ because zero BC for u at $y = 0$ and $y = H$ implies that $u \equiv 0$ ~~for~~ at ~~for~~ $t > 0$.

7.3.2 b

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$$u_t = k(u_{xx} + u_{yy} + u_{zz})$$

$0 < x < L$

$0 < y < H$

$0 < z < W$

$$\frac{\partial u}{\partial x}(0,y,z,t) = \frac{\partial u}{\partial x}(L,y,z,t) = \frac{\partial u}{\partial y}(x,0,z,t) = \frac{\partial u}{\partial y}(x,H,z,t) = \frac{\partial u}{\partial z}(x,y,0,t) = \frac{\partial u}{\partial z}(x,y,W,t) = 0$$

$$u(x,y,z,0) = \alpha(x,y,z)$$

$$u(x,y,z,t) = \phi(x,y,z)\, h(t)$$

$$\Rightarrow \frac{dh}{dt} = -\lambda k h$$

$$\nabla^2 \phi = -\lambda \phi$$

Extra separation of variables:

$$\phi(x,y) = f(x)\, g(y)\, Q(z)$$

$$\nabla^2 \phi + \lambda \phi = f_{xx} g Q + f g_{yy} Q + f g Q_{zz} + \lambda f g Q = 0$$

$$\Rightarrow \underset{-\mu}{\frac{f_{xx}}{f}} + \underset{-\nu}{\frac{g_{yy}}{g}} + \underset{\mu+\nu}{\frac{Q_{zz}}{Q}} + \lambda = 0$$

$$\Rightarrow \begin{cases} \dfrac{d^2 f}{dx^2} = -\mu f \\ f'(0) = f'(L) = 0 \end{cases} \Rightarrow \mu = \left(\frac{n\pi}{L}\right)^2$$

$$f = \cos\frac{n\pi x}{L}, \quad n = 0, 1, \ldots$$

$$\begin{cases} \dfrac{d^2 g}{dy^2} = -\nu g \\ g'(0) = g'(H) = 0 \end{cases} \Rightarrow \nu = \frac{m\pi}{H}$$

$$g = \cos\frac{m\pi y}{H}, \quad m = 0, 1, \ldots$$

$$\begin{cases} \dfrac{d^2 Q}{dz^2} = -(\lambda - \mu - \nu) Q \\ Q'(0) = Q'(W) = 0 \end{cases} \Rightarrow \lambda - \mu - \nu = \left(\frac{\ell\pi}{W}\right)^2$$

$$Q = \cos\left(\frac{\ell\pi z}{W}\right), \quad \ell = 0, 1, 2, \ldots$$

$$\Rightarrow \lambda_{nm\ell} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{\ell\pi}{W}\right)^2$$

$$\phi = \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos\left(\frac{\ell\pi z}{W}\right)$$

$$n, m, \ell = 0, 1, 2, \ldots$$

Superposition:

$$u = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} a_{nml}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{L}\right)\cos\left(\frac{l\pi z}{W}\right)e^{-\lambda_{nml}kt}$$

$$IC:\quad f = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} a_{nml}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{L}\right)\cos\left(\frac{l\pi z}{W}\right)$$

$\Rightarrow$ by orthogonality:

$$a_{nml} = \frac{\int_0^W\int_0^H\int_0^L \cancel{f}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos\left(\frac{l\pi z}{W}\right)\alpha(x,y,z)\,dx\,dy\,dz}{\int_0^W\int_0^H\int_0^L \cancel{f}\cos^2\left(\frac{n\pi x}{L}\right)\cos^2\left(\frac{m\pi y}{L}\right)\cos^2\frac{l\pi z}{W}\,dx\,dy\,dz}$$

As $t\to\infty$ all terms will decay but

$$A_{000} = \frac{1}{LHW}\int_0^L\int_0^H\int_0^W \alpha(x,y)\, f\,dx\,dy\,dz$$

$$\Rightarrow u(x,y,z,t)\Big|_{t\to\infty} = A_{000}$$

7.3.4.b

$$u_{tt} = c^2(u_{xx} + u_{yy})$$

$0<x<L$, $0<y<H$

IC: $u(x,y,0)=0$ $\quad \frac{\partial u}{\partial t}(x,y,0) = \alpha(x,y)$

BC: $\frac{\partial u}{\partial x}(0,y,t) = \frac{\partial u}{\partial x}(L,y,t) = 0$

$\frac{\partial u}{\partial y}(x,0,t) = \frac{\partial u}{\partial y}(x,H,t) = 0$

$$u = \phi(x,y) \, h(t)$$

$\frac{d^2 h}{dt^2} = -\lambda c^2 h. \Rightarrow h = c_1 \cos(\sqrt{\lambda} t) + c_2 \sin(\sqrt{\lambda} t)$

$h(0)=0 \Rightarrow \boxed{c_1 = 0 \text{ if } \lambda > 0 \text{ and } h = c_3 t \text{ if } \lambda = 0}$

Separation of variables for ϕ:

$$\phi = f(x)\, g(y)$$

$\Rightarrow \begin{cases} \frac{d^2 f}{dx^2} = -\mu f \\ f'(0) = f'(L) = 0 \end{cases}$ $\Rightarrow \mu = \left(\frac{n\pi}{L}\right)^2$

Eigenfunction

$f = \cos \frac{n\pi x}{L}$, $n = 0, 1, \ldots$

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$$\begin{cases} \dfrac{d^2 g}{dy^2} = -(\lambda - \mu) g \\ g'(0) = g'(H) = 0 \end{cases} \Rightarrow \lambda - \mu = \left(\frac{m\pi}{H}\right)^2$$

Eigenfunction

$$g = \cos\left(\frac{m\pi y}{H}\right), \quad m = 0, 1, \ldots$$

$$\Rightarrow \lambda_{nm} = \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}$$

Superposition:

$$\boxed{u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{nm} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) h_{nm}(t)}$$

Because of $h(0)=0$:

$$\boxed{h_{nm}(t) = \begin{cases} t & , n = m = 0 \\ \sin(\omega_{nm} t) & , \text{otherwise}, \end{cases}}$$

here $\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}$

Second IC:

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \alpha(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{nm} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) \cdot h'_{nm}(0)$$

$$\Rightarrow A_{nm} h'_{nm}(0) = \frac{\int_0^H \int_0^L f \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) dx\, dy}{\int_0^H \int_0^L \cos^2\left(\frac{n\pi x}{L}\right) \cos^2\left(\frac{m\pi y}{L}\right) dx\, dy}.$$

Note that $h'_{nm}(0) = 1$ for $n = m = 0$

but otherwise $h'_{nm}(0) = \omega_{nm}$

7.3.7 c

$$u_{xx} + u_{yy} + u_{zz} = 0$$

$$0 < x < L, \quad 0 < y < W, \quad 0 < z < H$$

$$\frac{\partial u}{\partial x}(0, y, z) = 0$$

$$\frac{\partial u}{\partial x}(L, y, z) = \alpha(x, y, z)$$

$$\frac{\partial u}{\partial y}(x, 0, z) = 0$$

$$\frac{\partial u}{\partial y}(x, W, z) = 0$$

$$\frac{\partial u}{\partial z}(x, y, 0) = \frac{\partial u}{\partial z}(x, y, H) = 0$$

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$u = h(x)\,\Phi(y,z) \Rightarrow$

$\dfrac{d^2 h}{dx^2} = \beta h$ and $\boxed{\begin{array}{l} \nabla^2\Phi + \beta\Phi = 0 \\ \dfrac{\partial\Phi}{\partial y}(0,z) = \dfrac{\partial\Phi}{\partial y}(W,z) = 0 \\ \dfrac{\partial\Phi}{\partial z}(y,0) = \dfrac{\partial\Phi}{\partial z}(y,H) = 0 \end{array}}$

$\Rightarrow h = \cosh(\sqrt{\beta}\,x)$
because $h'(0) = 0$

sol. is: $\phi = f(y)\,g(z)$

From 7.3.7.(b):

$$\begin{cases} \dfrac{d^2 f}{dy^2} = -\mu f \\ f'(0) = f'(W) = 0 \end{cases} \qquad \mu = \left(\frac{n\pi}{W}\right)^2$$

Eigenfunction:
$f = \cos\dfrac{n\pi y}{W}, \; n = 0, 1, \ldots$

$$\begin{cases} \dfrac{d^2 g}{dz^2} = -(\lambda - \mu)\, g \\ g'(0) = g'(H) = 0 \end{cases} \Rightarrow \lambda - \mu = \left(\frac{m\pi}{H}\right)^2$$

Eigenfunction

$g = \cos\left(\dfrac{m\pi z}{H}\right), \; m = 0, 1, \ldots$

$$\boxed{\beta_{nm}^2 = \left(\frac{n\pi}{W}\right)^2 + \left(\frac{m\pi}{H}\right)^2}$$

Superposition:

$$\boxed{u(x,y,z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{nm} \cos\left(\frac{n\pi y}{W}\right) \cos\left(\frac{m\pi z}{H}\right) \cosh(\beta_{nm} x)}$$

Nonhomogeneous boundary condition:

$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = \alpha(y,z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{nm} \beta_{nm} \cancel{\sinh\left(\frac{n\pi y}{W}\right) \cos\left(\frac{m\pi z}{H}\right)}$$

$$\times \sinh(\beta_{nm} L) \cos\left(\frac{n\pi y}{W}\right) \cos\left(\frac{m\pi z}{H}\right)$$

~~At~~

$\Rightarrow$ orthogonality:

$$\boxed{A_{nm} \sinh(\beta_{nm} L) = \frac{\int_0^H \int_0^W \alpha(y,z) \cos\left(\frac{n\pi y}{W}\right) \cos\left(\frac{m\pi z}{H}\right) dy\, dz}{\int_0^H \int_0^W \cos^2 \frac{n\pi y}{W} \cos^2 \frac{m\pi z}{H}\, dy\, dz}}$$

At $n=m=0$: $\sinh(0 \cdot L)=0$

$\Rightarrow$ require $\boxed{\int_0^H \int_0^W \alpha(y,z)\, dy\, dz = 0}$

& $\Rightarrow$ A_{00} - arbitrary.

7.5.6

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$$I = \iint_R \left[u L(v) - v L(u) \right] dx\, dy$$

$$L = \nabla^2 + q(x,y)$$

$$L(u) = \nabla^2 u + q u$$

$$L(v) = \nabla^2 v + q v$$

$$\Rightarrow \quad u L(v) - v L(u) = u \nabla^2 v + \cancel{q u v} - v \nabla^2 u - \cancel{q v u}$$

$$= u \nabla^2 v - v \nabla^2 u$$

$$I = \iint_R \left[u L(v) - v L(u) \right] dx\, dy = \iint_R (u \nabla^2 v - v \nabla^2 u)\, dx\, dy$$

$$= \oint_{\Gamma} (u \vec{\nabla} v - v \vec{\nabla} u) \cdot \hat{n}\, ds$$

boundary of R