HW02 solutions math 312

AT

2.3.2.b

$$(*) \quad \frac{d^2\Phi}{dx^2} + \lambda\Phi = 0$$

$$\Phi(0) = 0, \quad \Phi(1) = 0$$

General sol of $(*)$:

$\underline{\lambda > 0}$
$$\Phi = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

$$\Phi(0) = c_1 = 0$$
$$\Phi(1) = c_2 \sin(\sqrt{\lambda}) = 0$$
$$\Rightarrow \sqrt{\lambda} = \pi n, \quad \lambda = n^2\pi^2, \quad n = 1, 2, \ldots$$

$\underline{\lambda = 0}$
$$\Phi = c_1 + c_2 x$$
$$\Phi(0) = c_1 = 0$$
$$\Phi(1) = c_2 = 0 \quad \Rightarrow \text{ trivial sol.}$$

$\underline{\lambda < 0}$

A2

$\underline{\lambda < 0}$

$$\phi = c_1 e^{\sqrt{-\lambda} x} + c_2 e^{-\sqrt{-\lambda} x}$$

$$\phi(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$\phi(1) = c_1 \underbrace{\left(e^{\sqrt{-\lambda}\, 1} - e^{-\sqrt{-\lambda}\, 1}\right)}_{>0} \Rightarrow c_1 = 0$$

$\Rightarrow \phi \equiv 0$ - trivial sol only.

I.e. only solutions
for $\lambda = n^2\pi^2$, $n = 1, 2, \ldots$

2.3.2.d

$$\phi(0) = 0, \quad \frac{d\phi}{dx}(L) = 0$$

$\underline{\lambda > 0}$ $\quad \phi = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$

$\phi(0) = 0 \Rightarrow c_1$

A3

$$\frac{d\varphi(L)}{dx} = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)\Big|_{x=L} = 0$$

$$\Rightarrow \sqrt{\lambda}L = n\pi - \frac{\pi}{2}, \quad n=1,2,\ldots$$

$$\lambda = \frac{\left(n\pi - \frac{\pi}{2}\right)^2}{L^2}, \quad n=1,2,\ldots$$

$\underline{\lambda = 0}$

$$\Rightarrow \varphi = c_1 + c_2 x$$

$$\varphi(0) = c_1 = 0$$

$$\frac{d\varphi(L)}{dL} = c_2 = 0 \Rightarrow \text{trivial sol.}$$

$\lambda = 0$ is Not eigenvalue

$\underline{\lambda < 0}$

$$\varphi = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

$$\varphi(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$\frac{d\varphi(L)}{dL} = \sqrt{-\lambda}\left(c_1 e^{\sqrt{-\lambda}L} + c_1 e^{-\sqrt{-\lambda}L}\right)$$

$$= c_1\sqrt{-\lambda}\underbrace{\left(e^{\sqrt{-\lambda}L} + e^{-\sqrt{-\lambda}L}\right)}_{\neq 0} = 0 \Rightarrow c_1 = 0$$

A4

$\Rightarrow$ only trivial solution $c_1 = c_2 = 0$

$\Rightarrow$ $\lambda < 0$ - Not the eigenvalue

HW 02 Solutions

math 312

2.3.3 b

$$u_t = k u_{xx}$$
$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = 3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)$$

From (2.3.30) of textbook:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

IC: $$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = 3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)$$

$\Rightarrow$ by visual inspection $B_1 = 3$, $B_3 = -1$,

$B_2 = B_4 = B_5 = \ldots 0$

Or by (2.3.32):

$$B_n = \frac{2}{L}\int_0^L \left(3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

~~W/ by eqns from Wiki~~

$$\cancel{= \frac{2}{L}\int_0^L \frac{3}{2}\left[\cos\left((1-n)\frac{\pi x}{L}\right) - \cos\left((1+n)\frac{\pi x}{L}\right)\right]dx}$$

$$= \frac{2}{L} \cdot 3 \cdot \frac{L}{2} \delta_{n,1} + \frac{2}{L}(-1)\frac{L}{2}\delta_{n,3}$$

$$= 3\delta_{n,1} - 1\,\delta_{n,3} \quad , \text{ i.e. } B_1 = 3,\ B_3 = -1,\ B_2 = B_4 = \ldots 0$$

$$\boxed{u(x,t) = 3\sin\left(\frac{\pi x}{L}\right) e^{-k\left(\frac{\pi}{L}\right)^2 t} - \sin\left(\frac{3\pi x}{L}\right) e^{-9k\left(\frac{\pi}{L}\right)^2 t}}$$

2.3.3. c $\quad u(x,0) = 2\cos\frac{3\pi x}{L}$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n = \frac{2}{L}\int_0^L 2\cos\left(\frac{3\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \left\{ \text{by } \cos\theta \cdot \sin\varphi = \frac{1}{2}\left[\sin(\theta+\varphi) - \sin(\theta-\varphi)\right] \text{ from Wiki} \right\}$$

$$= \frac{2}{L}\,\frac{2}{2}\int_0^L \left(\sin\frac{(3+n)\pi x}{L} - \sin\frac{(3-n)\pi x}{L}\right) dx$$

$$= \frac{2}{L}\left[-\frac{L}{\pi(3+n)}\cos\left(\frac{(3+n)\pi x}{L}\right)\cdot\left(\underbrace{\delta_{n+3,0}}_{\text{0 because } n\geq 1}+1\right) + \frac{L}{\pi(3-n)}\cos\left(\frac{\pi x}{L}\right)\left(\delta_{3-n,0}+1\right)\right]_{x=0}^{L}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^{3+n}-1}{3+n}(-1) + \frac{1}{3-n}\left[(-1)^{3-n}-1\right]\left(1-\delta_{3-n,0}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n+1}{3+n} + \frac{(-1)^{n+1}}{n-3}\underbrace{\left(1-\delta_{3-n,0}\right)}_{=\,0 \text{ if } n=3}\right]$$

$$= \frac{4}{\pi}\begin{cases} 0, & n \text{ odd} \\ \frac{1}{3+n} - \frac{1}{n-3}, & n \text{ even} \end{cases}$$

2.3.4 a

$$u_t = k\frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = u(L,t) = 0, \quad u(x,0) = f(x)$$

$$\text{total heat energy} = \int_0^L c\rho u A \, dx$$

$$= c\rho A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \int_0^L \sin\frac{n\pi x}{L}\, dx$$

$$= c\rho A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \frac{-\cos(n\pi)+1}{\frac{n\pi}{L}}$$

$$= c\rho A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \frac{1-(-1)^n}{\frac{n\pi}{L}},$$

where $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\, dx$

according to (2.3.35)

2.3.4. b

Heat flux to the right $= -K_0 \dfrac{\partial u}{\partial x}$

Total heat flow to the right $= -K_0 \dfrac{\partial u}{\partial x} A$

At $x=0$: heat flow out $= K_0 \dfrac{\partial u}{\partial x}\Big|_{x=0} A$

$$= K_0 A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \frac{n\pi}{L} \underbrace{\cos\left(\frac{n\pi x}{L}\right)\Big|_{x=0}}_{=1}$$

$$= K_0 A \sum_{n=1} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \frac{n\pi}{L}$$

At $x=L$ heat flow out $= -K_0 \dfrac{\partial u}{\partial x}\Big|_{x=L} A$

$$= -K_0 A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \frac{n\pi}{L} \underbrace{\cos(n\pi)}_{=(-1)^n}.$$

2.3.4.c From conservation of thermal energy

$$\underbrace{\frac{d}{dt}\int_0^L u\,dx}_{\text{rate of change of thermal energy}} = k\frac{\partial u}{\partial x}\Big|_0^L = \underbrace{k\frac{\partial u}{\partial x}(L) - k\frac{\partial u}{\partial x}(0)}_{\text{fluxes through ends}}$$

$$k = \frac{K_0}{c\rho}$$

2.3.5

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_0^L\left(\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right)dx$$

$$= \frac{1}{2}\left[\delta_{n,m}L + (1-\delta_{n,m})\frac{\sin\frac{(n-m)\pi x}{L}}{\frac{(n-m)\pi}{L}}\Big|_0^L\right.$$

$$\left. - \underbrace{\delta_{n,-m}}_{=0}L + \underbrace{(1-\delta_{n,-m})}_{=0}\frac{\sin\frac{(n+m)\pi x}{L}}{\frac{n-m\,\pi}{L}}\Big|_0^L\right]$$

since $n, m > 0$

$$= \frac{L}{2}\delta_{n,m}$$

2.3.8

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} - \alpha u \quad , \alpha > 0 , \kappa > 0$$

$$u(0,t) = u(L,t) = 0.$$

(a) Equilibrium sol:

$$\kappa \frac{d^2 u}{dx^2} = \alpha u$$

General ODE solution:

$$u(x) = a \cosh\left(\sqrt{\tfrac{\alpha}{\kappa}}x\right) + b \sinh\left(\sqrt{\tfrac{\alpha}{\kappa}}\, x\right)$$

$$u(0) = a = 0$$
$$u(L) = b \sinh\left(\sqrt{\tfrac{\alpha}{\kappa}}\, L\right) = 0$$
$$\Rightarrow b = 0$$

$$\Rightarrow \boxed{u(x) \equiv 0}$$ – only equilibrium.

(b) separation of variables:

$$u = \Phi(x) \cdot h(t)$$

$$\Rightarrow \Phi \frac{dh}{dt} + \alpha \Phi h = \kappa h \frac{d^2\Phi}{dx^2}$$

$$\frac{\frac{dh}{dt}}{\kappa h} + \frac{\alpha}{\kappa} = \frac{1}{\Phi}\frac{d^2\Phi}{dx^2} = -\lambda$$

ODE BVP:

$$\begin{cases} \dfrac{d^2\Phi}{dx^2} = -\lambda \Phi \\ \Phi(0) = \Phi(L) = 0 \end{cases} \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \; n = 1, 2, \ldots$$

Eigenvalues:

Eigenfunctions:

$$\Phi = \sin\left(\frac{n\pi x}{L}\right)$$

For h:

$$\frac{dh}{dt} + \alpha h = -\kappa\lambda h := -\kappa\left(\frac{n\pi}{L}\right)^2 h$$

$$\Rightarrow h = c\, e^{-\alpha t} \cdot e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t}$$

Superposition:

$$\boxed{u(x,t) = e^{-\alpha t} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t}}$$

IC: $u(x,0) = f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

orthogonality $\Rightarrow \boxed{b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)}$.

As $t \to \infty \Rightarrow u \to 0$ as in equilibrium in (a).

2.4.1 a $\quad u_t = k u_{xx}, \quad 0 < x < L, \quad t > 0.$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, \quad t > 0$$

In text (2.4.19) u is found as:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

with (2.4.23-24)

$$\begin{cases} A_0 = \frac{1}{L} \int_0^L f(x)\,dx \\ A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \end{cases}$$

$\Rightarrow$ (a) $\quad u(x,0) = \begin{cases} 0, & x < \frac{L}{2} \\ 1, & x > \frac{L}{2} \end{cases}$

$$A_0 = \frac{1}{L} \int_{\frac{L}{2}}^{L} dx = \frac{1}{2}$$

$$A_n = \frac{2}{L} \int_{L/2}^{L} \cos \frac{n\pi x}{L}\, dx = \frac{2}{L} \cdot \frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)\Big|_{x=\frac{L}{2}}^{L}$$

$$= \frac{2}{n\pi} \Big(\underbrace{\sin(n\pi)}_{=0} - \sin \frac{n\pi}{2} \Big) = -\frac{2}{n\pi} \begin{cases} 0, & n = 2,4,6, \dots \\ 1, & n = 1,5,9, \dots \\ -1, & n = 3,7,11, \dots \end{cases}$$

2.4.1 (b) $u(x,0) = 6 + 4\cos\frac{3\pi x}{L}$

by inspection: $A_0 = 6$, $A_3 = 4$,

others $= 0$.

2.4.1 (c) $u = -2\sin\left(\frac{\pi x}{L}\right)$

$$A_0 = \frac{1}{L}\int_0^L -2\sin\frac{\pi x}{L}\,dx = \frac{2}{\pi}\cos\frac{\pi x}{L}\Big|_0^L = \frac{2}{\pi}(-1+\cos\pi)$$

$$= -\frac{4}{\pi}$$

$$A_n = -\frac{4}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx$$

$\{$ use $\sin\theta\cos\varphi = \frac{1}{2}\left[\sin(\theta+\varphi) + \sin(\theta-\varphi)\right]\}$

$$= -\frac{2}{L}\int_0^L \left[\sin\left(\frac{(n+1)\pi x}{L}\right) + \sin\left(\frac{(n-1)\pi x}{L}\right)\right]dx$$

$$= \frac{2}{L}\frac{L}{(n+1)\pi}\cos\left(\frac{(n+1)\pi x}{L}\right)\Big|_0^L - \frac{2}{L}\frac{L}{(n-1)\pi}\cos\frac{(n-1)\pi x}{L}\Big|_0^L \cdot \delta_{n,1}$$

$$= \frac{2}{(n+1)\pi}\left[(-1)^{n+1} - 1\right] - \frac{2}{(n-1)\pi}\left[(-1)^{n-1} - 1\right]\delta_{n,1}$$

$$= \begin{cases} -\dfrac{4}{(n+1)\pi} - \dfrac{4}{(n-1)\pi}\,\delta_{n,1}\,, & n\text{-even} \\ 0, & n \text{ odd}, \end{cases}$$

2.4.2

$$u_t = k\,u_{xx}\,,$$

$$\frac{\partial u}{\partial x}(0,t)=0\,, \quad u(L,t)=0$$

$$u(x,0) = f(x)$$

$$u = \phi(x)\,h(t)$$

$$\phi \cdot \frac{dh}{dt} = k\,h\,\frac{d^2\phi}{dx^2}$$

$$\Rightarrow \quad \frac{\frac{dh}{dt}}{h} = k\,\frac{\frac{d^2\phi}{dx^2}}{\phi} = -\lambda$$

$$\frac{dh}{dt} = -\lambda h \quad \Rightarrow \quad h = c\,e^{-\lambda t}$$

$$\frac{d^2\phi}{dx^2} = -\frac{\lambda}{k}\phi \quad \Rightarrow \phi = A\cos\left(\sqrt{\frac{\lambda}{k}}\,x\right) + B\sin\left(\sqrt{\frac{\lambda}{k}}\,x\right)$$

$$\Phi'(0) = 0 \Rightarrow -A\sqrt{\tfrac{\lambda}{k}}\sin\left(\sqrt{\tfrac{\lambda}{k}}x\right) + B\sqrt{\tfrac{\lambda}{k}}\cos\left(\sqrt{\tfrac{\lambda}{k}}x\right)\Big|_{x=0}$$

$$= B\sqrt{\tfrac{\lambda}{k}} = 0 \Rightarrow B = 0$$

$$\Phi(L) = A\cos\left(\sqrt{\tfrac{\lambda}{k}}L\right) = 0$$

$$\Rightarrow \sqrt{\tfrac{\lambda}{k}}L = \left(n+\tfrac{1}{2}\right)\pi \quad , n = 0, 1, \ldots$$

$$\Rightarrow \lambda = \frac{\left(n+\frac{1}{2}\right)^2\pi^2}{L^2}k \quad \text{- eigenvalues}$$

Eigenfunctions: $\cos\left(\dfrac{\left(n+\frac{1}{2}\right)\pi}{L}x\right)$

Superposition:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left[\frac{\left(n+\frac{1}{2}\right)\pi x}{L}\right] e^{-\frac{\left(n+\frac{1}{2}\right)^2\pi^2}{L^2}kt}$$

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{\left(n+\frac{1}{2}\right)\pi x}{L}\right) = f(x)$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{\left(n+\frac{1}{2}\right)\pi x}{L}\,dx \quad , n = 0, 1, \ldots$$

2.4.3

$$\phi_{xx} = -\lambda \phi$$

$$\phi(0) = \phi(2\pi), \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi)$$

Set $y = x - \pi$

$$\Rightarrow \begin{cases} \phi_{yy} = -\lambda \phi \\ \phi(-\pi) = \phi(\pi) \\ \frac{d\phi(-\pi)}{dy} = \frac{d\phi(\pi)}{dy} \end{cases}$$

$L = \pi$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 = n^2,$$

$n = 0, 1, \ldots$

Eigenfunctions

$$\cos\left(\frac{n\pi y}{L}\right) = \cos(ny)$$

$$= \cos(n(x-\pi)) = (-1)^n \cos(nx)$$

$$\sin\left(\frac{n\pi y}{L}\right) = \sin(ny)$$

$$= \sin(n(x-\pi))$$

$$= (-1)^n \sin nx$$

$(-1)^n$ can be included into arbitrary constant $\Rightarrow$ eigenfunctions $\sin(nx)$, $\cos(nx)$, 2.5.1a

$$\Delta u = 0$$

$$0 \le x \le L, \quad 0 \le y \le H$$

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad \frac{\partial u}{\partial x}(L,y) = 0,$$

$$u(x,0) = 0, \quad u(x,H) = f(x)$$

$$u = h(x)\,\phi(y)$$

$$\Rightarrow \quad \frac{\frac{d^2h}{dx^2}}{h} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} = -\lambda$$

$$\Rightarrow \begin{cases} \frac{d^2h}{dx^2} = -\lambda h \\ h'(0) = 0 \\ h'(L) = 0 \end{cases} \Rightarrow \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \; n = 0, 1, 2, \ldots$$

$$h = \cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad \frac{d^2\phi}{dy^2} = \left(\frac{n\pi}{L}\right)^2 \phi$$

$\underline{n=0}$: $\phi = C_1 + C_2 y$, $\phi(0) = 0 \Rightarrow C_1 = 0$

$\underline{n \neq 0}$: $\Phi = c_1 \cosh\left(\frac{n\pi y}{L}\right) + c_2 \sinh\left(\frac{n\pi y}{L}\right)$

$$\Phi(0) = c_1 = 0$$

Superposition:

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$$

the nonhomogeneous BC:

$$f(x) = u(x,H) = A_0 H + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi H}{L}\right) \cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow A_0 H = \frac{1}{L} \int_0^L f(x)\, dx$$

$$A_n \sinh\left(\frac{n\pi H}{L}\right) = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

2.5.1.e

$$u(0,y) = u(L,y) = 0$$

$$u(x,0) = \frac{\partial u}{\partial y}(x,0)$$

$$u(x,H) = f(x)$$

$$u = \phi(x)\, h(y)$$

$$\frac{\frac{d^2\phi(x)}{dx^2}}{\phi} = -\frac{\frac{d^2h}{dy^2}}{h} = -\lambda$$

ODE BVP:

$$\begin{cases} \frac{d^2\phi}{dx^2} = -\lambda\phi \\ \phi(0) = \phi(L) = 0 \end{cases}$$

$\Rightarrow$ Eigenvalues $\lambda = \left(\frac{n\pi}{L}\right)^2$

Eigenfunctions:

$\phi = \sin\frac{n\pi x}{L}$, $n = 1, 2, \ldots$

$$\begin{cases} \frac{d^2h}{dy^2} = \left(\frac{n\pi}{L}\right)^2 h \\ h(0) = \frac{dh}{dy}(0) \end{cases}$$

$\Rightarrow$ general sol

$h = c_1\cosh\left(\frac{n\pi y}{L}\right) + c_2\sinh\left(\frac{n\pi y}{L}\right)$

$h(0) - \frac{dh}{dy}(0) = c_1 - \frac{n\pi}{L}\left[\sinh\left(\frac{n\pi y}{L}\right)\right.$

$$+ c_2 \cosh\left(\frac{n\pi y}{L}\right)\Big]\Big|_{y=0}$$

$$= c_1 - \frac{n\pi}{L} c_2 = 0 \Rightarrow c_1 = c_2 \frac{n\pi}{L}$$

$$\Rightarrow h = \cosh\left(\frac{n\pi y}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi y}{L}\right)$$

Superposition

$$u(x,y) = \sum_{n=1}^{\infty} A_n \left[\cosh\left(\frac{n\pi y}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi y}{L}\right)\right] \times \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots$$

where A_n is determined from

$$\text{BC} \quad f(x) = \sum_{n=1}^{\infty} A_n \left[\cosh\left(\frac{n\pi H}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi H}{L}\right)\right] \times \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow A_n \left[\cosh\left(\frac{n\pi H}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi H}{L}\right)\right] = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$