

Problem 1

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$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = 0$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

From class:

$$\frac{\partial}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right) = \cos^2 \theta \frac{\partial^2}{\partial r^2} + \frac{\cos \theta \sin \theta}{r^2} \frac{\partial}{\partial \theta} - \frac{\cos \theta \sin \theta}{r} \frac{\partial^2}{\partial r \partial \theta}$$

$$+ \sin^2 \theta \frac{\partial^2}{\partial r^2} - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2}{\partial \theta \partial r} + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2}{\partial \theta^2}$$

(1)

$$\frac{\partial}{\partial y} = \frac{\sin \theta}{r} \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} \right) = \frac{\sin^2 \theta}{r^2} \frac{\partial^2}{\partial r^2} - \frac{1}{r^2} \sin \theta \cos \theta \frac{\partial}{\partial \theta}$$

$$+ \sin^2 \theta \cos \theta \frac{\partial^2}{\partial r \partial \theta}$$

$$+ \cos^2 \theta \frac{\partial}{\partial r} + \cos \theta \sin \theta \frac{\partial^2}{\partial \theta \partial r} - \frac{\cos \theta \sin \theta}{r^2} \frac{\partial}{\partial \theta}$$

$$+ \cos^2 \theta \frac{\partial^2}{\partial \theta^2} \quad (2)$$

$$(1) + (2) = (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2}{\partial r^2} + (\sin^2 \theta + \cos^2 \theta) \frac{\partial}{\partial r}$$

$$+ (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2}{\partial \theta^2}$$

$$\Rightarrow \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} u \right) + \frac{1}{r} \frac{\partial^2 u}{\partial \theta^2} = 0 \right]$$

math 312

HW 03 Solutions

(7)

2.5.2a

$$0 < x < L$$
$$0 < y < H$$

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = 0$$

$$\frac{\partial u}{\partial x}(L, y) = 0, \quad \frac{\partial u}{\partial y}(x, H) = f(x)$$

(a) the total heat flow across the boundary must be equal to zero in equilibrium (i.e. without sources in Laplace  $\psi$ ).

$$\Rightarrow \int_0^L f(x) dx = 0$$

(b) using (2.5.16):  $u = h(x) \Phi(y)$

$$\frac{h_{xx}}{h} = -\frac{\Phi_{yy}}{\Phi} = -\lambda$$

$$\begin{cases} h_{xx} = -\lambda h \\ \frac{\partial h(0)}{\partial x} = \frac{\partial h(L)}{\partial x} = 0 \end{cases}$$

homogeneous BVP

$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, n \neq 1, 2, 3, \dots$

$h(x) = \frac{\cos \frac{n\pi x}{L}}{L}$  eigenfunctions

②

$$h = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$$

$$h_x = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} x) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x)$$

$$h_x(0) = c_2 \sqrt{\lambda} = 0 \Rightarrow c_2 = 0$$

$$h_x(L) = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} L) = 0$$

$$\Rightarrow \lambda = 0 \text{ or } \sqrt{\lambda} L = n\pi, n = 0, 1, \dots$$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, n = 0, 1, 2, \dots$$

Eigen functions:  $\cos\left(\frac{n\pi}{L} x\right)$

$$\lambda = 0: \quad h = c_1 + c_2 x$$

$$h_x = c_2 = 0$$

$$\begin{cases} \frac{d^2 \Phi}{dy^2} = \left(\frac{n\pi}{L}\right)^2 \Phi \\ \frac{d\Phi}{dy}(0) = 0 \end{cases}$$

$$\Rightarrow \Phi = \cosh\left(\frac{n\pi}{L} y\right)$$

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Superposition:

$$u(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right)$$

$$\frac{\partial u(x, H)}{\partial y} = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi H}{L}\right) = f(x)$$

$$\Rightarrow A_n \cosh\left(\frac{n\pi H}{L}\right) = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$n=1, 2, \dots$

$$u(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right)$$

(\*)

$A_0$  - arbitrary.

(optional) : to find  $A_0$

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u, \quad u(x, y, 0) = g(x, y)$$

Initial heat energy:  $E(0) = c\rho \int_0^L \int_0^H u(x, y, 0) dx dy$

$$= c\rho \int_0^L \int_0^H g(x, y) dx dy$$

rate of change of heat energy:

$$\frac{dE}{dt} = - \int_0^L k_0 \frac{\partial u}{\partial y}(x, H) dx = -k_0 \int_0^L f(x) dx$$

$= 0$  according to (a)

$$\Rightarrow E(t) \Big|_{t \rightarrow \infty} = E(0) = c_p \int_0^L \int_0^H g(x, y) dx dy \quad (4)$$

$$\text{But } E(\infty) = \int c_p \int_0^L \int_0^H u(x, y) dx dy$$

(using solution (x) of (b))

$$= c_p \cdot L \cdot H \cdot A_0 + 0$$

$$\Rightarrow A_0 = \frac{\int_0^L \int_0^H u(x, y) dx dy}{LH}$$

(2.5.3) outside of a circular disk  $r > a$



$$\nabla^2 u = \frac{1}{r} \partial_r (r \partial_r u) + \frac{1}{r^2} \partial_\theta^2 u = 0$$

$$u = \phi(\theta) G(r)$$

$$\frac{r}{G} \frac{d}{dr} \left( r \frac{dG}{dr} \right) = -\frac{1}{\phi} \frac{d^2 \phi}{d\theta^2} = 1$$

(5)

$$\left\{ \begin{aligned} \frac{d^2 \phi}{d\theta^2} &= -\lambda \phi \\ \phi(-\pi) &= \phi(\pi) \\ \frac{d\phi}{d\theta}(-\pi) &= \frac{d\phi}{d\theta}(\pi), \quad L = \pi \end{aligned} \right.$$

$$\phi(-\pi) = \phi(\pi)$$

$$\frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi), \quad L = \pi$$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 = n^2 - \text{eigenvalues}$$

$\sin(n\theta), \cos(n\theta)$  - eigenfunctions.

$$\frac{r}{G} \frac{d}{dr} \left( r \frac{dG}{dr} \right) = n^2$$

$$r^2 \frac{d^2 G}{dr^2} + r \frac{dG}{dr} - n^2 G = 0$$

$$n \neq 0: \quad G = c_1 r^n + c_2 r^{-n} \quad \left. \begin{array}{l} \text{to have } |u| < \infty \\ \text{as } r \rightarrow \infty \\ \Rightarrow c_1 = c_2 = 0 \end{array} \right\}$$

$$n = 0 \Rightarrow G = \bar{c}_1 + \bar{c}_2 \ln r$$

Superposition:

$$u = \sum_{n=0}^{\infty} A_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^{-n} \sin(n\theta)$$

$$(a) \quad u(a, \theta) = \ln 2 + 4 \cos(3\theta) \Rightarrow A_0 = \ln 2, \quad A_3 a^3 = 4, \quad A_n = B_n = 0 \text{ for other } n$$

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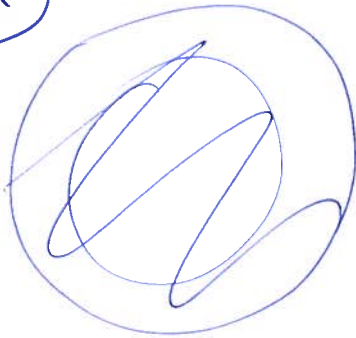
$$(b) \quad u(a, \theta) = f(\theta)$$

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$A_n a^{-n} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta$$

$$B_n a^{-n} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta$$

2.5.6.a



$$0 < \theta < \pi$$
$$0 < r < a$$

$$\nabla^2 u = 0$$

$u = 0$  at diameter and  $u(a, \theta) = g(\theta)$

Similar to (2.5.37) but different BC:

$$\begin{cases} \frac{d^2 \Phi}{d\theta^2} = -\lambda \Phi \\ \Phi(0) = \Phi(\pi) = 0 \end{cases}$$

$\Rightarrow \lambda = \left(\frac{n\pi}{\pi}\right)^2 = n^2$  eigenvalues  
 $\sin(n\theta)$ ,  $n = 1, 2, 3, \dots$   
- eigenfunctions.



$$Q = C_1 r^n + C_2 r^{-n}, \quad C_2 = 0 \text{ to have } |u| < \infty \text{ at } r=0 \quad (7)$$

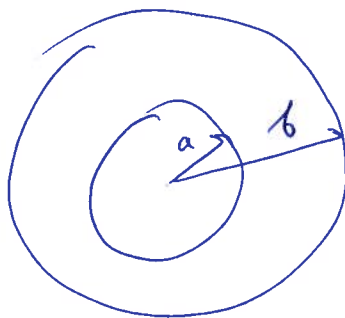
Superposition:

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n r^n \sin(n\theta)$$

BC at  $r=a$ :  $g(\theta) = \sum_{n=1}^{\infty} A_n a^n \sin(n\theta)$

$$\Rightarrow A_n a^n = \frac{2}{\pi} \int_0^{\pi} g(\theta) \sin(n\theta) d\theta$$

2.5.8 a



$$a < r < b$$

$$\nabla^2 u = 0$$

$$BC: u(a, \theta) = f(\theta)$$

$$u(b, \theta) = g(\theta)$$

$$u = \Phi(\theta) G(r) \Rightarrow$$

$$\nabla^2 u = \frac{1}{r} \partial_r (r \partial_r u) + \frac{1}{r^2} \partial_{\theta}^2 u = 0$$

$$\frac{r}{G} \frac{d}{dr} \left( r \frac{dG}{dr} \right) = - \frac{1}{\Phi} \frac{d^2 \Phi}{d\theta^2} = \lambda \quad -\pi < \theta < \pi$$

⑧

Periodicity in  $\varphi$ :

$$\begin{cases} \varphi(\bar{r}) = \varphi(-\bar{r}) \\ \frac{d\varphi(\bar{r})}{d\theta} = \frac{d\varphi(-\bar{r})}{d\theta} \\ \frac{d^2\varphi}{d\theta^2} = -\lambda\varphi \end{cases}$$

$$\Rightarrow \varphi = c_1 \cos(\sqrt{\lambda}\theta) + c_2 \sin(\sqrt{\lambda}\theta)$$

$$\frac{d\varphi}{d\theta} = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda}\theta) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda}\theta)$$

BC:  $c_1 \cos(\sqrt{\lambda}R) + c_2 \sin(\sqrt{\lambda}R) = c_1 \cos(\sqrt{\lambda}R) - c_2 \sin(\sqrt{\lambda}R)$   
 $- c_1 \sqrt{\lambda} \sin(\sqrt{\lambda}R) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda}R) = c_1 \sqrt{\lambda} \sin(\sqrt{\lambda}R) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda}R)$

$$\Rightarrow \sin(\sqrt{\lambda}R) = 0 \Rightarrow \sqrt{\lambda}R = nR, \quad n \in \mathbb{Z}$$

$$\lambda = n^2, \quad n = 0, 1, \dots - \text{eigenvalues}$$

~~NAP~~  $\cos(n\theta), \sin(n\theta) - \text{eigenfunctions}$

$$\lambda \neq 0 \Rightarrow \tilde{c}_1 + \tilde{c}_2 \ln r$$

$$\frac{r}{G} \frac{d}{dr} \left( r \frac{dG}{dr} \right) = n^2 \Rightarrow r^2 \frac{d^2 G}{dr^2} + r \frac{dG}{dr} - n^2 G = 0$$

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$$\underline{\lambda \neq 0} \quad G = C_1 r^n + C_2 r^{-n}, \quad n=1, 2, \dots$$

$$\underline{\lambda = 0} \quad G = \bar{c}_1 + \bar{c}_2 \ln r$$

It is convenient to choose instead  $r^{\pm n}$  the following eigenfunctions:

$$G_1(r) = \begin{cases} \ln \frac{r}{a}, & n=0 \\ \left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n, & n \neq 0 \end{cases} \Rightarrow G_1(a) = 0$$

$$G_2(r) = \begin{cases} \ln \left(\frac{r}{b}\right), & n=0 \\ \left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^n, & n \neq 0 \end{cases} \Rightarrow G_2(b) = 0$$

Superposition:

$$u(r, \theta) = \sum_{n=0}^{\infty} \cos(n\theta) [A_n G_1(r) + B_n G_2(r)]$$

$$+ \sum_{n=1}^{\infty} \sin(n\theta) [C_n G_1(r) + D_n G_2(r)]$$

$$\underline{BC} : r=a : f(\theta) = \sum_{n=0}^{\infty} \cos(n\theta) [B_n G_2(a)]$$

$$+ \sum_{n=1}^{\infty} \sin(n\theta) [D_n G_2(a)]$$

$$\underline{r=b} \quad g(\theta) = \sum_{n=0}^{\infty} \cos(n\theta) [A_n G_1(b)]$$

$$+ \sum_{n=1}^{\infty} \sin(n\theta) [C_n G_1(b)]$$

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By orthonormality

$$B_0 G_2(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$B_n G_2(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\theta) f(\theta) d\theta, \quad n=1, 2, \dots$$

$$A_0 G_1(b) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) d\theta$$

$$A_n G_1(b) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) \cos(n\theta) d\theta, \quad n=1, 2, \dots$$

$$C_0 G_1(b) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) d\theta$$

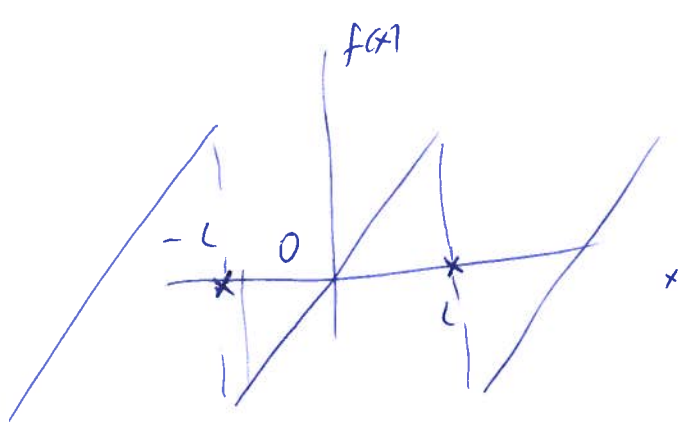
$$C_n G_1(b) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(n\theta) g(\theta) d\theta$$

$$D_0 G_2(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$D_n G_2(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(n\theta) f(\theta) d\theta$$

3.2.2.a

-L ≤ x ≤ L



$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{2L} \left. \frac{x^2}{2} \right|_{-L}^L = 0$$

$$a_n = \frac{1}{L} \int_{-L}^L x \cos\left(\frac{n\pi x}{2}\right) dx = 0 \quad \left. \begin{array}{l} \text{func} \\ x \text{ is odd!} \end{array} \right\}$$

$$b_n = \frac{2L}{\pi n} (-1)^{n+1} \quad (\text{from class})$$

$$\Rightarrow x \sim \sum_{n=1}^{\infty} \frac{2L}{\pi n} (-1)^{n+1} \sin\left(\frac{n\pi}{2} x\right)$$