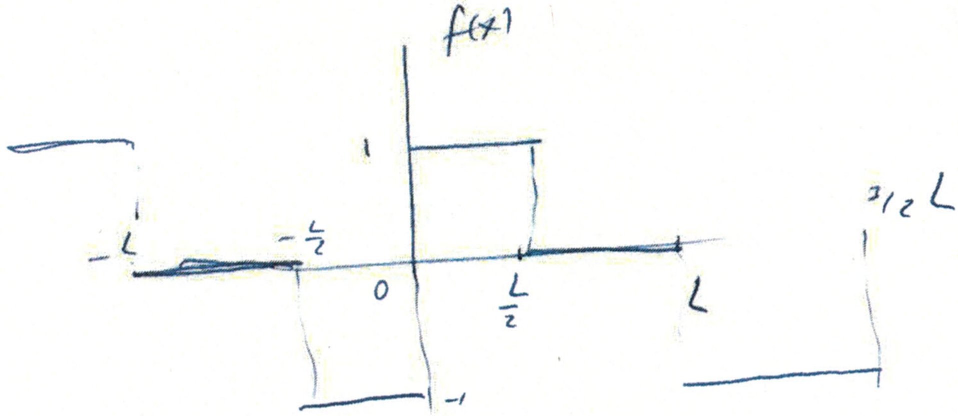
Math 312 HW 04

3.3.2 d

$$f(x) = \begin{cases} 1 & , x < \frac{L}{2} \\ 0 & , x > \frac{L}{2} \end{cases}$$

Odd extension



$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n}{L} x\right) dx$$

$$= \frac{2}{L} \int_0^{L/2} f(x) \sin\left(\frac{\pi n}{L} x\right) dx = \frac{2}{L} \frac{-L}{\pi n} \cos\left(\frac{n\pi}{L} x\right)\Big|_0^{L/2}$$

$$= \frac{2}{n\pi}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)$$

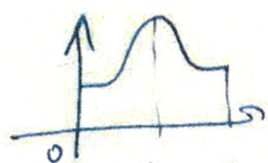
3.3.10

$$f(x) = \begin{cases} x^2, & x < 0 \\ e^{-x}, & x > 0 \end{cases} \Rightarrow f(-x) = \begin{cases} x^2, & -x < 0 \\ e^{x}, & -x > 0 \end{cases}$$

$$f_e(x) = \frac{1}{2}\left[f(x) + f(-x)\right] = \frac{1}{2}\begin{cases} x^2 + e^{x}, & x < 0 \\ x^2 + e^{-x}, & x > 0 \end{cases}$$

$$f_o(x) = \frac{1}{2}\left[f(x) - f(-x)\right] = \frac{1}{2}\begin{cases} x^2 - e^{x}, & x < 0 \\ -x^2 + e^{-x}, & x > 0 \end{cases}$$

3.3.13



$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

But $\sin\left(\frac{n\pi x}{L}\right)$ is odd around $\frac{L}{2}$ for n even

$\Rightarrow f(x) \sin\left(\frac{n\pi x}{L}\right)$ is odd around $\frac{L}{2}$ for

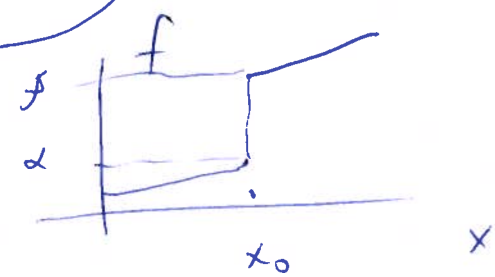
n even $\Rightarrow b_n = 0$ for n even.

HW 04 Solutions

math 312

(1)

3.4.3a



$$f \sim \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad a_n = \frac{2}{L}\int_0^L f \cos\frac{n\pi x}{L}\,dx$$
$n \neq 0$

We want to find:

$$f \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Use integration by parts:

$$b_n = \frac{2}{L}\int_0^L \frac{df}{dx}\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\,df$$

$$= \frac{2}{L}\left[f\sin\frac{n\pi x}{L}\Big|_0^{x_0-} + f\sin\frac{n\pi x}{L}\Big|_{x_0+}^{L} - \frac{n\pi}{L}\int_0^L f\cos\frac{n\pi x}{L}\,dx \right] = \frac{2}{L}\left[\alpha \sin\left(\frac{n\pi x_{0-}}{L}\right) - 0 \right.$$

$$\left. + 0 - \beta\sin\left(\frac{n\pi x_{0+}}{L}\right) - \frac{n\pi}{L}\frac{L}{2} a_n \right]$$

$$= \frac{2}{L}\sin\left(\frac{n\pi x_0}{L}\right)(\alpha-\beta) - \frac{n\pi}{L}a_n$$

3.4.9

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x,t)$$

$u(0,t)=0$

$u(L,t)=0$

$q(x,t)$ - piecewise smooth in x $\forall t$.

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{L} \qquad (*)$$

Because $\frac{\partial^2 u}{\partial x^2}$ is piecewise smooth and $u(0,t)=0=u(L,t)$

and $u(x,t)$ is continuous

$\Rightarrow$ we can differentiate $(*)$ term by term:

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)b_n\cos\left(\frac{n\pi x}{L}\right) \qquad (**)$$

Because $\frac{\partial^2 u}{\partial x^2}$ is piecewise smooth

$\Rightarrow$ we can differentiate $(**)$ term by term:

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 b_n \sin\frac{n\pi x}{L}$$

thus from p. 119 and from class:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{d b_n}{dt}\sin\frac{n\pi x}{L} \sim -k\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 b_n \sin\frac{n\pi x}{L} + q$$

$$\Rightarrow \boxed{\frac{d b_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n = \frac{2}{L}\int_0^L q(x,t)\sin\left(\frac{n\pi x}{L}\right)dx}$$

3.4.12

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + e^{-t} + e^{-2t}\cos\frac{3\pi x}{L}$$

(assume $2 \neq k\left(\frac{3\pi}{L}\right)^2$)

BC: $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$

IC: $u(x,0) = f(x)$.

the eigenfunctions of homogeneous ODE BVP ~~problem~~ are $\cos\frac{n\pi x}{L}$, $n = 0, 1, 2, \ldots$

$\Rightarrow \quad u \sim \sum_{n=0}^{\infty} A_n(t) \cos\left(\frac{n\pi x}{L}\right)$ – it can be differentiated (if u is continuous) since it is a cosine series:

$$\frac{\partial u}{\partial x} \sim \sum_{n=0}^{\infty} A_n \left(-\frac{n\pi}{L}\right) \sin \frac{n\pi x}{L}.$$

this can be differentiated again (if $\frac{\partial u}{\partial x}$ is continuous) only because

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$$ – according to BC

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} \sim -\sum_{n=0}^{\infty} A_n \left(\frac{n\pi}{L}\right)^2 \cos\left(\frac{n\pi x}{L}\right)$$

$\Rightarrow$ {similar to problem 3.4.9}:

$$\frac{\partial u}{\partial t} + k\frac{\partial^2 u}{\partial x^2} = \sum_{n=0}^{\infty} \left[\frac{dA_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 A_n\right] \cos\frac{n\pi x}{L} = \underbrace{e^{-t} + e^{-2t}\cos\left(\frac{3\pi x}{L}\right)}_{\text{cos series in } x}$$

$$(**) \quad \Rightarrow \quad \frac{dA_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 A_n = \begin{cases} e^{-t}, & n=0 \\ e^{-2t}, & n=3 \\ 0, & n \neq 0, n \neq 3 \end{cases}$$

IC gives $A_0(0) = \frac{1}{L}\int_0^L f(x)\,dx$

$$A_n(0) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{2}\right)dx, \quad n \neq 0$$

ODE (***) is solved as follows:

$\underline{n=0}$: $\frac{dA_0}{dt} = e^{-t} \Rightarrow A_0(t) - A_0(0) = \int_0^t e^{-t'}dt' = 1 - e^{-t}$

$$\boxed{A_0(t) = A_0(0) + 1 - e^{-t}}$$

$n \neq 0, n \neq 3$: $\frac{dA_n}{dt} = -\kappa\left(\frac{n\pi}{2}\right)^2 A_n$

$$\Rightarrow \boxed{A_n = A_n(0)\, e^{-\kappa\left(\frac{n\pi}{2}\right)^2 t}}$$

$\underline{n=3}$ $A_3 = c(t)\, e^{-\kappa\left(\frac{3\pi}{2}\right)^2 t}$

$$\Rightarrow \frac{dc}{dt}\, e^{-\kappa\left(\frac{3\pi}{2}\right)^2 t} = e^{-2t}$$

$$c = \underbrace{c(0)}_{= A_3(0)} + \int_0^t e^{-2t' + \kappa\left(\frac{3\pi}{2}\right)^2 t'}\,dt' = A_3(0)$$

$$+ \frac{e^{-2t + \kappa\left(\frac{3\pi}{2}\right)^2 t} - 1}{-2 + \kappa\left(\frac{3\pi}{2}\right)^2}$$

$$\Rightarrow A_3 = A_3(0)\, e^{-\kappa\left(\frac{3\pi}{2}\right)^2 t} + \frac{e^{-2t} - e^{-\kappa\left(\frac{3\pi}{2}\right)^2 t}}{-2 + \kappa\left(\frac{3\pi}{2}\right)^2}$$

3.5.1 c

$$x^2 \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$\int_0^x dx\,[\cdots] \Rightarrow \quad \frac{x^3}{3} = \sum_{n=1}^{\infty} b_n \left(\frac{L}{n\pi}\right)\left(-\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^x$$

$$= \sum_{n=1}^{\infty} b_n\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n\left(\frac{L}{n\pi}\right)$$

~~Also according to (3.5.11)~~

$$\cancel{A_0 = \sum_{n=1}^{\infty} \frac{b_n}{n\pi/L} = \frac{1}{L}\int_0^L x^2\,dx = \frac{L^2}{3}}$$

7

3.5.7. Evaluate (3.5.6) at $x = \frac{L}{2}$:

$$\left.\frac{x^2}{2}\right|_{\frac{L}{2}} = \frac{L^2}{8} = \left.\frac{L}{2}x - \frac{4L^2}{\pi^3}\sum_{n\,odd}\frac{\sin\left(\frac{n\pi x}{L}\right)}{n^3}\right|_{x=\frac{L}{2}}$$

$$= \frac{L^2}{4} - \frac{4L^2}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \dots\right)$$

$$\Rightarrow \boxed{1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots} = -\frac{L^2}{8}\cdot\left(-\frac{\pi^3}{4L^2}\right)$$

$$= \boxed{\frac{\pi^3}{32}}$$

3.6.1

$$f(x) = \begin{cases} 0, & x < x_0 \\ \frac{1}{\Delta} & x_0 < x < x_0 + \Delta \\ 0, & x > x_0 + \Delta \end{cases}$$

$\Rightarrow$ using (3.6.7)

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{\frac{in\pi x}{L}}\,dx = \frac{1}{2L\Delta}\int_{x_0}^{x_0+\Delta} e^{\frac{in\pi x}{L}}\,dx.$$

⑧

$$= \frac{1}{2L\Delta}\frac{L}{in\pi} e^{\frac{inx\pi}{L}} \Bigg|_{x_0}^{x_0+\Delta} = \boxed{\frac{1}{2in\pi\Delta} e^{\frac{in\pi x_0}{L}} \left(e^{\frac{in\pi\Delta}{L}} - 1\right) = c_n}$$

(equivalently)

$$\boxed{= \frac{1}{n\pi\Delta} e^{in\pi\left(x_0 + \frac{\Delta}{2}\right)} \sin\frac{n\pi\Delta}{2L}}$$