math 312 test 2 solutions

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1. (40 points) Solve the damped string equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}, \quad 0 < x < L, \ \beta > 0, \ \beta^2 < (2c\pi/L)^2,$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0,$$

and the initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

For a full credit you have to perform separation of variables (you can be brief in description of the separation of variables but must be clear in explaining how you do that), obtain eigenvalues and eigenfunctions, analyze for which sign of the eigenvalue you have solution, derive the formula for the general solution by superposition and satisfy the initial condition. You are also allowed to save much by writing explicitly the eigenvalues and eigenfunctions of ODE boundary value problem and shortly referring to the class (instead of deriving them). But make sure you are precise in identifying correct eigenvalues and eigenfunctions, otherwise you will receive a zero credit for that part of the problem. In contrast, if you choose to derive eigenvalues and eigenfunctions, you will receive a partial credit even if you make some part of that derivation incorrectly. Also you do not need to show the orthogonality of eigenfunctions and you do not need to evaluate integrals. You can also refer to other problem in the test if you already used the same ODE boundary value problem there.

$$u = \phi(x)\, h(t)$$

$$\frac{\frac{d^2 h}{dt^2}}{h} = \frac{c^2 \frac{d^2\phi}{dx^2}}{\phi} - \beta \frac{\frac{dh}{dt}}{h}$$

$$\frac{c^2 \frac{d^2\phi}{dx^2}}{\phi} = -\lambda$$

$\Rightarrow$ ODE BVP

$$\begin{cases} \frac{d^2\phi}{dx^2} = -\lambda\phi \\ \phi(0) = \phi(L) = 0 \end{cases} \Rightarrow \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots \qquad \phi = \sin\left(\frac{n\pi x}{L}\right)$$

ODE for $h(t)$: $h'' + \beta h' + \lambda c^2 h = 0$ ②

$$h = e^{rt}$$

$\to$ characteristic eq $r^2 + \beta r + \left(\frac{n\pi}{2}\right)^2 c^2 = 0$

$$\Rightarrow r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4\left(\frac{n\pi}{2}\right)^2 c^2}}{2}$$

Because $\lambda > 0$, i.e. $n = 1, 2, \ldots$ $\Rightarrow$ $\beta^2 - 4\left(\frac{n\pi}{2}\right)^2 c^2 < 0$

$$\Rightarrow r_{1,2} = -\frac{\beta}{2} \pm i\sqrt{c^2\left(\frac{n\pi}{2}\right)^2 - \frac{\beta^2}{4}}$$

General sol:

$$h = c_1 e^{-\frac{\beta}{2}t} \cos\left(\sqrt{c^2\left(\frac{n\pi}{2}\right)^2 - \frac{\beta^2}{4}}\, t\right) + c_2 e^{-\beta\frac{t}{2}} \sin\left(\sqrt{c^2\left(\frac{n\pi}{2}\right)^2 - \frac{\beta^2}{4}}\, t\right)$$

Superposition:

$$u(x,t) = e^{-\beta\frac{t}{2}} \sum_{n=1}^{\infty}\left[A_n \cos\left(\sqrt{c^2\left(\frac{n\pi}{2}\right)^2 - \frac{\beta^2}{4}}\, t\right) + B_n \sin\left(\sqrt{c^2\left(\frac{n\pi}{2}\right)^2 - \frac{\beta^2}{4}}\, t\right)\right] \sin\left(\frac{n\pi x}{2}\right)$$

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$$IC_s: \quad u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \qquad (*)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = -\frac{\beta}{2}\left(\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$f(x)$

$$+ \sqrt{c^2\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4}} \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad (**)$$

Orthogonality for $(*)$:

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} A_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

$(**)$

$$= A_m \frac{L}{2}$$

$$\Rightarrow \quad A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Orthogonality for $(**)$:

$$\int_0^L g(x) \sin\left(\frac{m\pi x}{L}\right) dx = -\frac{\beta}{2}\underbrace{\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx}_{=\frac{L}{2}A_n} +$$

$$+ \sum_{n=1}^{\infty} \sqrt{c^2\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4}} \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) B_n \, dx}_{=\delta_{n,m}\frac{L}{2}B_n} \qquad (***)$$

$$\Rightarrow \boxed{B_n = \frac{2}{L}\,\frac{\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx}{\sqrt{c^2\left(\frac{n\pi}{L}\right)^2-\frac{\beta^2}{4}}} + \frac{\beta A_n}{\sqrt{c^2\left(\frac{n\pi}{L}\right)^2-\frac{\beta^2}{4}}}}$$

2. (15 points) Consider the following Ordinary Differential Equation (ODE):

$$\beta(x)\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + \lambda\xi(x)\phi + \delta(x)\phi = 0,\ \alpha(x) \neq 0. \tag{1}$$

Bring ODE (1) to the standard Sturm-Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\phi = 0 \tag{2}$$

through multiplying (1) by the appropriate function of x. Find $p(x)$, $\sigma(x)$ and $q(x)$ assuming $\alpha(x)$, $\beta(x)$, $\xi(x)$, and $\delta(x)$ are given functions.

Multiplying (1) by a new function $H(x)$:

$$H(x)\cdot(1) = \beta H\phi'' + \alpha H\phi' + \lambda\xi H\phi + \delta H\phi = 0$$

$$(2) = p\phi'' + p'\phi' + \lambda\sigma\phi + q\phi = 0$$

$$\beta H = p \Rightarrow H = \frac{p}{\beta} \quad p' = \alpha H = \frac{\alpha p}{\beta} \Rightarrow \frac{dp}{p} = \frac{\alpha}{\beta}dx$$

$$\boxed{p = ce^{\int^x \frac{\alpha(x')}{\beta(x')}dx'}}$$

$$\Rightarrow \boxed{H = \frac{p}{\beta} = \frac{c}{\beta}e^{\int^x \frac{\alpha(x')dx'}{\beta(x')}}}$$

Also

$$\boxed{\sigma = \xi H = \xi\frac{c}{\beta}e^{\int^x \frac{\alpha(x')dx'}{\beta(x')}}}$$

$$\boxed{q = H\delta = \delta c e^{\int^x \frac{\alpha(x')}{\beta(x')}dx'}}$$

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3. (45 points) Consider the heat equation in a three-dimensional box-shaped region, $0 < x < L$, $0 < y < H$, $0 < z < W$:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$$

subject to boundary and initial conditions:

$$\begin{aligned}
&u(x, y, 0, t) = u(x, y, W, t) = 0,\\
&\frac{\partial u}{\partial x}(0, y, z, t) = \frac{\partial u}{\partial x}(L, y, z, t) = 0,\\
&\frac{\partial u}{\partial y}(x, 0, z, t) = \frac{\partial u}{\partial y}(x, H, z, t) = 0,\\
&u(x, y, z, 0) = \alpha(x, y, z).
\end{aligned}$$

a) (5 points) Obtain equilibrium temperature distribution without solution of differential equation.
b) (40 points) Solve the initial value problem and compare with part (a).

For a full credit you have to perform separation of variables (you can be brief in description of the separation of variables but must be clear in explaining how you do that), obtain eigenvalues and eigenfunctions, analyze for which sign of the eigenvalue you have solution, derive the formula for the general solution by superposition and satisfy the initial condition. You are also allowed to save much by writing explicitly the eigenvalues and eigenfunctions of ODE boundary value problem and shortly referring to the class (instead of deriving them). But make sure you are precise in identifying correct eigenvalues and eigenfunctions, otherwise you will receive a zero credit for that part of the problem. In contrast, if you choose to derive eigenvalues and eigenfunctions, you will receive a partial credit even if you make some part of that derivation incorrectly. Also you do not need to show the orthogonality of eigenfunctions and you do not need to evaluate integrals. You can also refer to other problem in the test if you already used the same ODE boundary value problem there.

(a) we have no sources $\Rightarrow$ the only possible equilibrium sol is const.

But $u(x, y, 0, t) = 0 \Rightarrow$ const $= 0$

②

$$u(x,y,z,t) = \Phi(x,y,z)\, h(t)$$

$$= f(x)\, g(y)\, Q(z)\, h(t)$$

$$\underbrace{\frac{\frac{d^2 f}{dx^2}}{f}}_{= -\mu} + \underbrace{\frac{\frac{d^2 g}{dy^2}}{g}}_{= -\nu} + \underbrace{\frac{\frac{d^2 Q}{dz^2}}{Q}}_{= \mu+\nu-\lambda} = \underbrace{\frac{\frac{dh}{dt}}{\kappa h}}_{= -\lambda}$$

$$\frac{dh}{dt} = -\kappa h \lambda \Rightarrow h = c_1 e^{-\kappa \lambda t}$$

$$\begin{cases} \frac{d^2 f}{dx^2} = -\mu f \\ f'(0) = f'(L) = 0 \end{cases} \Rightarrow$$ Eigenvalues $\mu = \left(\frac{n\pi}{L}\right)^2,\ n = 0, 1, \ldots$

Eigenfunctions $\cos\left(\frac{n\pi x}{L}\right)$

$$\begin{cases} \frac{d^2 g}{dy^2} = -\nu g \\ g'(0) = g'(H) = 0 \end{cases} \Rightarrow$$ Eigenvalues $\nu = \left(\frac{m\pi}{H}\right)^2$

Eigenfunctions $\cos\left(\frac{m\pi y}{H}\right)$

$m = 0, 1, \ldots$

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$$\begin{cases} \dfrac{d^2Q}{dz^2} = -(\lambda - \mu - \nu)Q \\ Q(0) = Q(W) = 0 \end{cases} \Rightarrow$$

Eigenvalues $\lambda - \mu - \nu = \left(\frac{l\pi}{W}\right)^2$

Eigenfunctions $\sin\left(\frac{l\pi z}{W}\right)$

$l = 1, 2, \dots$

$$\Rightarrow \lambda_{nml} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{l\pi}{W}\right)^2$$

$l = 1, 2, \dots, \quad n, m = 0, 1, \dots$

Superposition; $\quad \Phi_{nml} = \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\sin\left(\frac{l\pi z}{W}\right)$

$$u = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=1}^{\infty} a_{nml} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\sin\left(\frac{l\pi z}{W}\right) e^{-\kappa \lambda_{nml} t}$$

$$IC: \alpha(x,y,z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=1}^{\infty} a_{nml} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\sin\left(\frac{l\pi z}{W}\right)$$

$$\Rightarrow a_{nml} = \frac{\int_0^W \int_0^H \int_0^L \alpha(x,y,z) \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\sin\left(\frac{l\pi z}{W}\right) dx\,dy\,dz}{\int_0^W \int_0^H \int_0^L \cos^2\left(\frac{n\pi x}{L}\right)\cos^2\left(\frac{m\pi y}{H}\right)\sin^2\left(\frac{l\pi z}{W}\right) dx\,dy\,dz}$$

$$\| \; \left(\frac{L}{2}\right)^3$$

For $t \to \infty$ $e^{-\lambda_{nml}\kappa t} \to 0$ the same result as in (a)