

Conservation law of chemical concentration:

1.2.5

$$\frac{d}{dt} \int_x^{x+\Delta x} u(x',t) dx' = \Phi(x,t) - \Phi(x+\Delta x,t) + \int_x^{x+\Delta x} \alpha u(x',t) (\beta - u(x',t)) dx'$$

$\Delta x \rightarrow 0$, $\Phi(x,t) \equiv$ flux.

$$\Rightarrow \frac{\partial}{\partial t} u(x,t) \Delta x = - \frac{\partial \Phi(x,t)}{\partial x} \Delta x + \alpha u(x,t) (\beta - u(x,t)) \Delta x$$

But $\Phi = - \kappa \frac{\partial u}{\partial x}$

$$\Rightarrow \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + \alpha u (\beta - u)$$

1.3.2

Next flux:

$$\Phi = -K_0(x) \frac{\partial u}{\partial x}$$

flux is continuous at $x=0$

which means that $\Phi(x_{0-}, t) = \Phi(x_{0+}, t)$

$$\Rightarrow K_0(x_{0-}) \frac{\partial u}{\partial x}(x_{0-}, t) = K_0(x_{0+}) \frac{\partial u}{\partial x}(x_{0+}, t)$$

Also if $k_0(x_0^-) = k_0(x_0^+)$ ②

i.e. $k_0(x)$ is continuous at $x=x_0$

$$\Rightarrow \frac{\partial u}{\partial x}(x_0^-, t) = \frac{\partial u}{\partial x}(x_0^+, t),$$

i.e. $\frac{\partial u}{\partial x}$ is continuous at $x=x_0$

1.4.1a $Q=0, u(0)=0, u(L)=T$

using eq (1.4.17)

$$u(x) = T_1 + T_2 = \frac{T_1}{L} x = \frac{T}{L} x$$

1.4.1c $Q=0, \frac{\partial u(0)}{\partial x} = 0, u(L)=T$

General steady-state sol:

$$\frac{d^2 u}{dx^2} = 0 \Rightarrow u(x) = c_1 x + c_2$$

$$\frac{\partial u(0)}{\partial x} = c_1 = 0$$

$$u(L) = 0 \cdot L + c_2 = T \Rightarrow$$

$u(x) = T$

1.4.1f

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$$\frac{Q}{k_0} = x^2, \quad u(0) = T, \quad \frac{\partial u}{\partial x}(L) = 0$$

General steady-state sol.

$$\frac{d}{dx} \left(k_0 \frac{\partial u}{\partial x} \right) + Q = 0$$

Because thermal properties of rod are constant $\Rightarrow k_0 = \text{const.}$

$$k_0 \frac{d^2 u}{dx^2} + x^2 k_0 = 0$$

$$\frac{d^2 u}{dx^2} + x^2 = 0$$

General sol of nonhomogeneous problem is obtained by twice integration over x :

$$u = -\frac{x^4}{12} + c_1 + c_2 x$$

To satisfy BC:

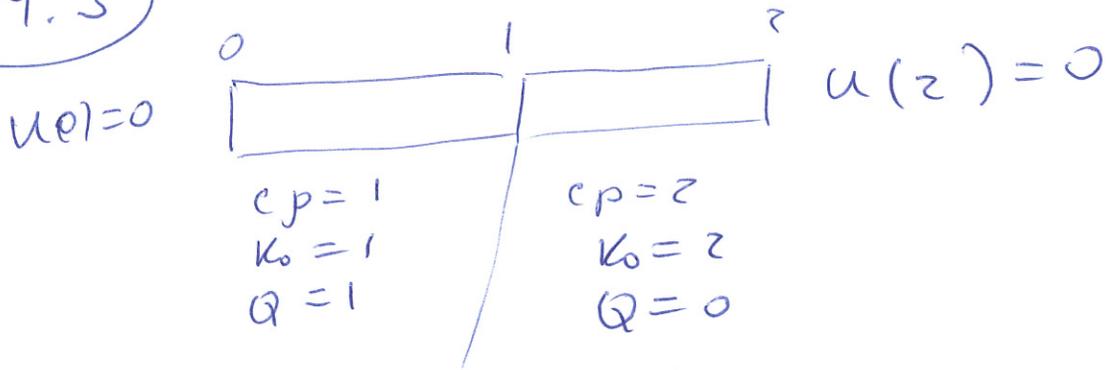
$$u(0) = c_1 = T$$

$$\frac{\partial u}{\partial x}(L) = -\frac{x^3}{3} + c_2 \Big|_{x=L} = -\frac{L^3}{3} + c_2 = 0$$
$$\Rightarrow c_2 = \frac{L^3}{3}$$

(9)

$$\Rightarrow u = -\frac{x^4}{12} + \frac{L^3}{3}x + T$$

1.4.3



According to 1, 3, 2, it implies

$$(1) \begin{cases} u(1_+, t) = u(1_-, t) \\ k_0(1_-) \frac{\partial u}{\partial x}(1_-, t) = k_0(1_+) \frac{\partial u}{\partial x}(1_+, t) \end{cases}$$

In rod 1:

$$1 \cdot \frac{d^2 u}{dx^2} + 1 = 0 \Rightarrow u = -\frac{x^2}{2} + c_1 + c_2 x$$

$$u(0) = c_1 = 0$$

$$(2) \begin{cases} \frac{\partial u}{\partial x}(1_+) = -x + c_2 \Big|_{x=1} = -1 + c_2 \\ u(1_+, t) = -\frac{1}{2} + c_2 \end{cases}$$

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In rod 2:

$$2 \cdot \frac{d^2 u}{dx^2} = 0 \Rightarrow u = C_3 + C_4 x$$

$$u(2) = C_3 + 2C_4 = 0 \Rightarrow C_3 = -2C_4$$

$$(3) \begin{cases} \frac{\partial u(1+, t)}{\partial x} = C_4 \\ u(1+, t) = -2C_4 + C_4 = -C_4 \end{cases}$$

Eqs. (1)-(3) has 6 unknowns $C_2, C_4, u(1+, t), u(1-, t), \frac{\partial u(1+, t)}{\partial x}, \frac{\partial u(1-, t)}{\partial x}$ and total 6 conditions.

So Excluding $u(1\pm, t)$ and $\frac{\partial u}{\partial x}(1\pm, t)$

We obtain:

$$\begin{cases} -1 + C_2 = 2C_4 & \Rightarrow C_2 = \frac{2}{3} \\ -\frac{1}{2} + C_2 = -C_4 & C_4 = -\frac{1}{6}, C_3 = -2C_4 = \frac{1}{3} \end{cases}$$

$$\Rightarrow \boxed{u = -\frac{x^2}{2} + \frac{2}{3}x, \quad 0 < x < 1}$$

$$\boxed{u = \frac{1}{3} - \frac{x}{6}, \quad 1 < x < 2}$$

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1.4.10

$$u_t = u_{xx} + 4 \quad \Rightarrow \quad \begin{matrix} k=1 \\ c_p=1 \end{matrix}$$

$$u(x, 0) = f(x)$$

$$\frac{\partial u}{\partial x}(0, t) = 5$$

$$\frac{\partial u}{\partial x}(L, t) = 6$$

Find total heat and energy $E(t)$ in a rod.

$$E = \int_0^L u(x, t) dx$$

Using eq (1.2.7):

$$\frac{dE}{dt} = \int_0^L \frac{\partial u}{\partial t} dx = \int_0^L \left(-\frac{\partial \Phi}{\partial x} + \underbrace{4}_{\text{Q}} \right) dx$$

$$= \Phi(0) - \Phi(L) + 4L$$

$$\text{But } \Phi(x) = -\frac{\partial u}{\partial x}$$

$$\Rightarrow \Phi(0) = -5$$

$$\Phi(L) = -6$$

$$\Rightarrow \frac{dE}{dt} = -5 + 6 + 4L = 4L + 1, \text{ also } E(0) = \int_0^L f(x) dx$$

$$\Rightarrow E(t) = \int_0^L f(x) dx + (4L + 1)t$$

2.3.1a

$$u_t = \frac{\kappa}{r} \partial_r (r \partial_r u)$$

$$u = \Phi(r) G(t)$$

$$\Rightarrow \Phi \frac{dG}{dt} = \frac{\kappa G}{r} \partial_r (r \partial_r \Phi)$$

$$\Rightarrow \frac{\frac{dG}{dt}}{\kappa G} = \frac{1}{r \Phi} \partial_r (r \partial_r \Phi) = -\lambda$$

$$\Rightarrow \begin{cases} \frac{dG}{dt} = -\kappa \lambda G \\ \frac{1}{r} \frac{d}{dr} (r \frac{d\Phi}{dr}) = -\lambda \Phi \end{cases}$$

2.3.1c

$$u_{xx} + u_{yy} = 0$$

$$u = \Phi(x) h(y)$$

$$\Rightarrow h \Phi_{xx} + \Phi h_{yy} = 0$$

dividing by Φh :

$$\frac{\Phi_{xx}}{\Phi} + \frac{h_{yy}}{h} = -\lambda \Rightarrow \begin{cases} \frac{d^2 \Phi}{dx^2} = -\lambda \Phi \\ \frac{d^2 h}{dy^2} = \lambda h \end{cases}$$

2.3.2.6

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$$(*) \quad \frac{d^2 \Phi}{dx^2} + \lambda \Phi = 0$$

$$\Phi(0) = 0, \quad \Phi(1) = 0$$

General sol of (*):

$$\underline{\lambda > 0} \quad \Phi = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$$

$$\Phi(0) = C_1 = 0$$

$$\Phi(1) = C_2 \sin(\sqrt{\lambda}) = 0$$

$$\Rightarrow \sqrt{\lambda} = n\pi, \quad \lambda = n^2 \pi^2, \quad n=1, 2, \dots$$

$$\underline{\lambda = 0}$$

$$\Phi = C_1 + C_2 x$$

$$\Phi(0) = C_1 = 0$$

$$\Phi(1) = C_2 = 0$$

\Rightarrow trivial sol.

Done

$\lambda < 0$

$$\phi = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}$$

$$\phi(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$\phi(1) = c_1 (e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}) \Rightarrow c_1 = 0$$

$\Rightarrow \phi \equiv 0$ - trivial sol only.

I.e. only solutions

or $k = n^2 \pi^2, n = 1, 2, \dots$

2.3.2.d

$$\phi(0) = 0, \quad \frac{d\phi}{dx}(L) = 0$$

$\lambda > 0$ $\phi = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$

$$\phi(0) = 0 \Rightarrow c_1$$

$$\frac{d\phi(L)}{dx} = c_2 \sqrt{\lambda} \cos(\sqrt{\lambda}x) \Big|_{x=L} = 0$$

$$\Rightarrow \sqrt{\lambda} L = n\pi - \frac{\pi}{2}, \quad n=1, 2, \dots$$

$$\lambda = \frac{\left(n\pi - \frac{\pi}{2}\right)^2}{L^2}, \quad n=1, 2, \dots$$

$$\underline{\lambda = 0} \Rightarrow \phi = c_1 + c_2 x$$

$$\phi(0) = c_1 = 0$$

$$\frac{d\phi(L)}{dx} = c_2 = 0 \Rightarrow \text{trivial sol.}$$

$\lambda = 0$ is Not eigenvalue

$$\underline{\lambda < 0} \quad \phi = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

$$\phi(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$\frac{d\phi(L)}{dx} = \sqrt{-\lambda} \left(c_1 e^{\sqrt{-\lambda}L} + c_1 e^{-\sqrt{-\lambda}L} \right)$$

$$= c_1 \sqrt{-\lambda} \left(e^{\sqrt{-\lambda}L} + e^{-\sqrt{-\lambda}L} \right) = 0 \Rightarrow c_1 = 0$$

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\Rightarrow only trivial solution $r_1 = r_2 = 0$

$\Rightarrow \lambda < 0$ - not the eigenvalue