math 312

# HW 03 Solutions

## 2.5.2a

$$0 < x < L$$
$$0 < y < H$$

$$\frac{\partial u}{\partial x}(0, y) = 0 \quad , \quad \frac{\partial u}{\partial y}(x, 0) = 0$$

$$\frac{\partial u}{\partial x}(L, y) = 0 \quad , \quad \frac{\partial u}{\partial y}(x, H) = f(x)$$

(a) the total heat flow across the boundary must be equal to zero in equilibrium (i.e. without sources in Laplace eq).

$$\Rightarrow \quad \int_0^L f(x)\,dx = 0$$

(b) using (2.5.16): $\quad u = h(x)\,\Phi(y)$

$$\frac{h_{xx}}{h} = -\frac{\Phi_{yy}}{\Phi} = -\lambda$$

$$\begin{cases} h_{xx} = -\lambda h \\ \frac{\partial h}{\partial x}(0) = \frac{\partial h}{\partial x}(L) = 0 \end{cases}$$

homogeneous BVP

~~$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, 3, \ldots$~~

~~$h(x) = \cos \frac{n\pi}{L} x$ eigenfunctions~~

$$h = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

$$h_x = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda}\, x) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda}\, x)$$

$$h_x(0) = c_2 \sqrt{\lambda} = 0 \Rightarrow c_2 = 0$$

$$h_x(L) = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda}\, L) = 0$$

$$\Rightarrow \lambda = 0 \text{ or } \sqrt{\lambda}\, L = n\pi, \; n = 0, 1 \ldots$$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2 \ldots$$

Eigenfunctions: $\cos\left(\frac{n\pi}{L} x\right)$

$\lambda = 0$ : $h = c_1 + c_2 x$

$h_x = c_2 = 0$

$$\begin{cases} \dfrac{d^2 \varphi}{dy^2} = \left(\dfrac{n\pi}{L}\right)^2 \varphi \\ \dfrac{d\varphi}{dy}(0) = 0 \end{cases}$$

$$\Rightarrow \varphi = \cancel{\sinh}\ \cosh\left(\frac{n\pi}{L} y\right)$$

Superposition:

$$u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right)$$

$$\frac{\partial u(x,H)}{\partial y} = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi H}{L}\right) = f(x)$$

$$\Rightarrow A_n \cosh\left(\frac{n\pi H}{L}\right) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad n=1,2,\ldots$$

$$u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right),$$

(*) $A_0$ – arbitrary.

((c (optional)) : to find $A_0$

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u, \qquad u(x,y,0) = g(x,y)$$

Initial heat energy: $E(0) = c\rho \int_0^L \int_0^H u(x,y,0)\,dx\,dy$

$$= c\rho \int_0^L \int_0^H g(x,y)\,dx\,dy$$

Rate of change of heat energy: $\dfrac{dE}{dt} = -\int_0^L K_0 \dfrac{\partial u}{\partial y}(x,H)\,dx = -K_0 \int_0^L f(x)\,dx$

$= 0$ according to (a)

$$\Rightarrow E(t)\Big|_{t\to\infty} = E(0) = c\rho \int_0^L \int_0^H g(x,y)\,dx\,dy$$

But $E(\infty) = c\rho \int_0^L \int_0^H u(x,y)\,dx\,dy$ (using solution $(*)$ of (8))

$$= c\rho \cdot L \cdot H \cdot A_0 + 0$$

$$\Rightarrow A_0 = \frac{\int_0^L \int_0^H u(x,y)\,dx\,dy}{LH}$$

2.5.3 outside of a circular disk $r \geq a$

$$\nabla^2 u = \frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_\theta^2 u = 0$$

$$u = \phi(\theta)\, G(r)$$

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda$$

$$\begin{cases} \dfrac{d^2\Phi}{d\theta^2} = -\lambda \Phi \\ \Phi(-\pi) = \Phi(\pi) \\ \dfrac{d\Phi}{d\theta}(-\pi) = \dfrac{d\Phi}{d\theta}(\pi) \end{cases}, \quad l = \pi$$

$\Rightarrow \lambda = \left(\frac{n\pi}{l}\right)^2 = n^2$ - eigenvalues

$\sin(n\theta)$, $\cos(n\theta)$ - eigenfunctions.

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = n^2$$

$$r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2 G = 0$$

$n \neq 0$: $\quad G = c_1 r^n + c_2 r^{-n}$

$n = 0 \Rightarrow G = \bar{c}_1 + \bar{c}_2 \ln r$

} to have $|u| < \infty$ as $r \to \infty$ $\Rightarrow c_1 = \bar{c}_2 = 0$

Superposition:

$$u = \sum_{n=0}^{\infty} A_n r^{-n} \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^{-n} \sin(n\theta)$$

(a) $u(a,\theta) = \ln 2 + 4\cos(3\theta)$

$\Rightarrow A_0 = \ln 2$, $A_3 a^{-3} = 4$, $A_n = B_n = 0$ for other $n$

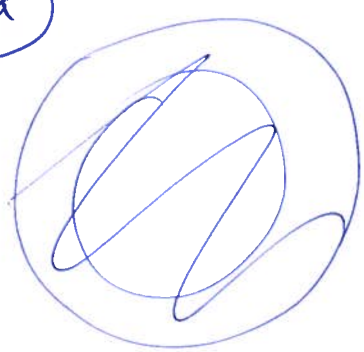
(b) $u(a,\theta) = f(\theta)$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$A_n a^{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$B_n a^{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta$$

2.5.6.a




$0 < \theta < \pi$

$0 < r < a$

$$\nabla^2 u = 0$$

$u = 0$ at diameter and $u(a,\theta) = g(\theta)$

Similar to (2.5.37) but different BC:

$$\begin{cases} \dfrac{d^2\Phi}{d\theta^2} = -\lambda\Phi \\ \Phi(0) = \Phi(\pi) = 0 \end{cases} \Rightarrow \lambda = \left(\frac{n\pi}{\pi}\right)^2 = n^2 \text{ eigenvalues}$$

$\sin(n\theta)$, $n = 1, 2, 3 \ldots$ – eigenfunctions.

$$G = c_1 r^n + c_2 r^{-n}, \quad c_2 = 0 \text{ to have } |u| < \infty \text{ at } r = 0$$

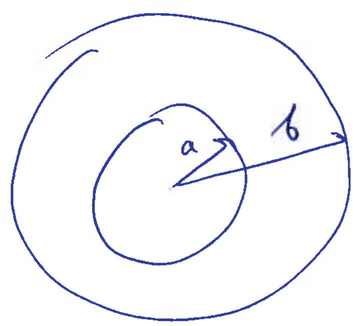
Superposition:

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin(n\theta)}$$

BC at $r = a$: $$g(\theta) = \sum_{n=1}^{\infty} A_n a^n \sin(n\theta)$$

$$\Rightarrow \boxed{A_n a^n = \frac{2}{\pi} \int_0^{\pi} g(\theta) \sin(n\theta)\, d\theta}$$

(2.5.8a)




$a < r < b$

$\nabla^2 u = 0$

BC: $u(a,\theta) = f(\theta)$

$u(b,\theta) = g(\theta)$

$$u = \Phi(\theta) G(r) \Rightarrow$$

$$\nabla^2 u = \frac{1}{r} \partial_r (r \partial_r u) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{r}{G} \frac{d}{dr}\left(r \frac{dG}{dr}\right) = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\theta^2} = \lambda$$

$-\pi < \theta < \pi$

Periodicity in $\varphi$: $\begin{cases} \varphi(\pi) = \varphi(-\pi) \\ \dfrac{d\varphi}{d\theta}(\pi) = \dfrac{d\varphi}{d\theta}(-\pi) \\ \dfrac{d^2\varphi}{d\theta^2} = -\lambda\varphi \end{cases}$

$$\Rightarrow \quad \varphi = c_1 \cos(\sqrt{\lambda}\,\theta) + c_2 \sin(\sqrt{\lambda}\,\theta)$$

$$\frac{d\varphi}{d\theta} = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,\theta) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,\theta)$$

BC: $\cancel{c_1 \cos(\sqrt{\lambda}\,\pi)} + c_2 \sin(\sqrt{\lambda}\,\pi) = \cancel{c_1 \cos(\sqrt{\lambda}\,\pi)} - c_2 \sin(\sqrt{\lambda}\,\pi)$

$$-c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi) + \cancel{c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi)} = c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi) + \cancel{c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi)}$$

$$\Rightarrow \sin(\sqrt{\lambda}\,\pi) = 0 \Rightarrow \sqrt{\lambda}\,\pi = n\pi, \quad n \in \mathbb{Z}$$

$\lambda = n^2$, $n = 0, 1, \ldots$ - eigenvalues

$\cancel{\lambda \neq 0}$ $\quad \cos(n\theta)$, $\sin(n\theta)$ - eigenfunctions

$\cancel{\lambda = 0 \Rightarrow \bar{c}_1 + \bar{c}_2 \ln r}$

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = n^2 \Rightarrow r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2 G = 0$$

$\underline{\lambda \neq 0}$ $G = c_1 r^n + c_2 r^{-n}$, $n = 1, 2, \dots$

$\underline{\lambda = 0}$ $G = \bar{c}_1 + \bar{c}_2 \ln r$

It is convenient to choose instead $r^{\pm n}$ the following eigenfunctions:

$$G_1(r) = \begin{cases} \ln \frac{r}{a}, & n = 0 \\ \left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n, & n \neq 0 \end{cases} \Rightarrow G_1(a) = 0$$

$$G_2(r) = \begin{cases} \ln \left(\frac{r}{b}\right), & n = 0 \\ \left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^n, & n \neq 0 \end{cases} \Rightarrow G_2(b) = 0$$

Superposition:

$$u(r, \theta) = \sum_{n=0}^{\infty} \cos(n\theta) \left[ A_n \cancel{\Phi_1} G_1(r) + B_n G_2(r) \right]$$

$$+ \sum_{n=1}^{\infty} \sin(n\theta) \left[ C_n G_1(r) + D_n \Phi_2(r) \right]$$

$\underline{BC}$ : $r = a$ : $$f(\theta) = \sum_{n=0}^{\infty} \cos(n\theta) \left[ B_n \cancel{\Phi_2} G_2(a) \right]$$

$$+ \sum_{n=1}^{\infty} \sin(n\theta) \left[ D_n \cancel{\Phi_2} G_2(a) \right]$$

$\underline{r = b}$ $$g(\theta) = \sum_{n=0}^{\infty} \cos(n\theta) \left[ A_n G_1(b) \right]$$

$$+ \sum \sin(n\theta) \left[ C_n G_1(b) \right]$$

$\Rightarrow$ By orthogonality

$$B_0\, G_2(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta.$$

$$B_n\, G_2(a) = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(n\theta)\, f(\theta)\, d\theta \quad , n = 1, 2, \ldots$$

$$A_0\, G_1(b) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, d\theta$$

$$A_n\, G_1(b) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\, \cos(n\theta)\, d\theta \quad , n = 1, 2, \ldots$$

$$C_0\, G_1(b) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, d\theta$$

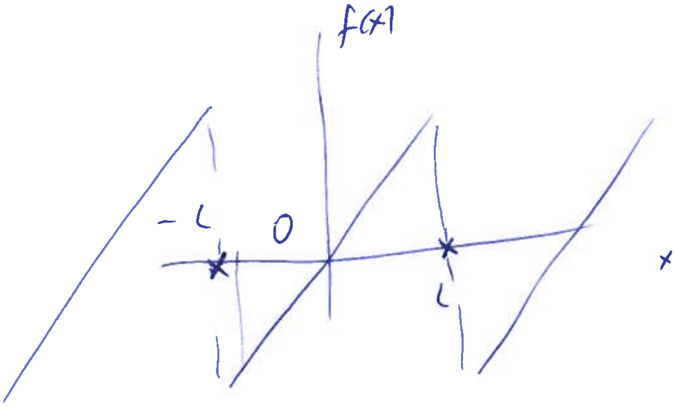
$$C_n\, G_1(b) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\theta)\, g(\theta)\, d\theta$$

$$D_0\, G_2(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$D_n\, G_2(a) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\theta)\, f(\theta)\, d\theta$$

3.2.2.a

$-L \le x \le L$




$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\frac{x^2}{2}\Big|_{-L}^{L} = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx = 0$$ since x is odd.

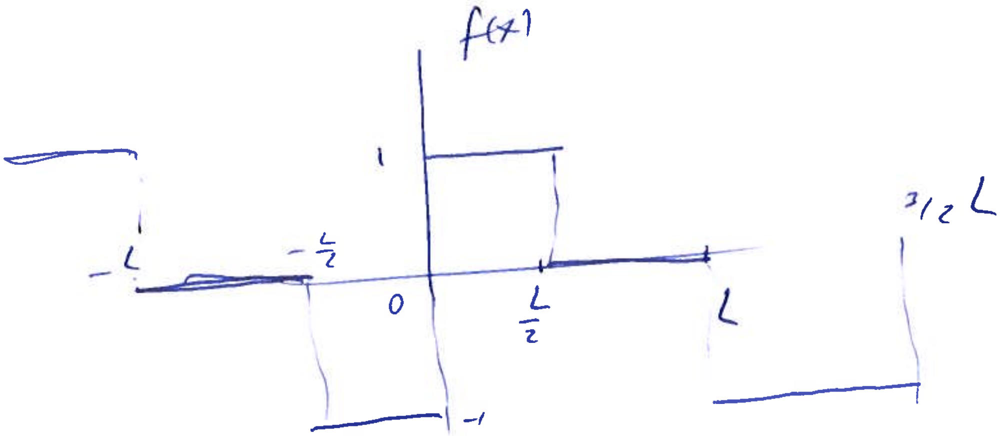
$$b_n = \frac{2L}{\pi n}(-1)^{n+1}$$ (from class)

$$\Rightarrow \quad x \sim \sum_{n=1}^{\infty} \frac{2L}{\pi n}(-1)^{n+1}\sin\left(\frac{n\pi}{L}x\right).$$

3.3.2d

$$f(x) = \begin{cases} 1 & , x < \frac{L}{2} \\ 0 & , x > \frac{L}{2} \end{cases}$$

Odd extension




$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n}{L} x\right) dx$$

$$= \frac{2}{L} \int_0^{L/2} f(x) \sin\left(\frac{\pi n x}{L}\right) dx = \frac{2}{L} \left(-\frac{L}{\pi n}\right) \cos\left(\frac{n\pi}{L} x\right)\Big|_0^{L/2}$$

$$= \frac{2}{n\pi} \left(1 - \cos\left(\frac{n\pi}{2}\right)\right)$$

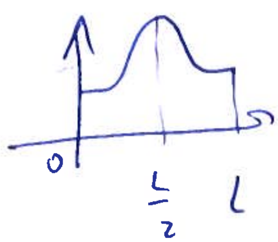
3.3.10

$$f(x) = \begin{cases} x^2, & x<0 \\ e^{-x}, & x>0 \end{cases} \Rightarrow f(-x) = \begin{cases} x^2, & -x<0 \\ e^{x} & -x>0 \end{cases}$$

$$f_e(x) = \frac{1}{2}\left[f(x)+f(-x)\right] = \frac{1}{2}\begin{cases} x^2+e^{x}, & x<0 \\ x^2+e^{-x}, & x>0 \end{cases}$$

$$f_o(x) = \frac{1}{2}\left[f(x)-f(-x)\right] = \frac{1}{2}\begin{cases} x^2-e^{x}, & x<0 \\ -x^2+e^{-x}, & x>0 \end{cases}$$

3.3.13




$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

But $\sin\left(\frac{n\pi x}{L}\right)$ is odd around $\frac{L}{2}$ for $n$ even

$\Rightarrow f(x)\sin\left(\frac{n\pi x}{L}\right)$ is odd around $\frac{L}{2}$ for $n$ even ~~and~~ $\Rightarrow b_n = 0$ for $n$ even.