Practice for test 1 (excerpts from hws)

~~Also if $K_0(x_{0-}) = K_0(x_{0+})$~~

~~i.e. $K_0(x)$ is continuous at $x=x_0$~~

~~$\Rightarrow \frac{\partial u}{\partial x}(x_{0-},t) = \frac{\partial u}{\partial x}(x_{0+},t)$~~

~~i.e. $\frac{\partial u}{\partial x}$ is continuous at $x=x_0$~~

1.4.1a $Q=0$, $u(0)=0$, $u(L)=T$

using eq (1.4.17)

$$u(x) = T_1 + \frac{T_2 - T_1}{L}x = \frac{T}{L}x$$

1.4.1c $Q=0$ $\frac{\partial u}{\partial x}(0)=0$, $u(L)=T$

General steady-state sol:

$$\frac{d^2u}{dx^2}=0 \Rightarrow u(x) = C_1x + C_2$$

$$\frac{\partial u(0)}{\partial x} = C_1 = 0$$

$$u(L) = 0x + C_2 = T \Rightarrow \boxed{u(x) = T}$$

1.4.1 f

$$\frac{Q}{K_0} = x^2, \quad u(0) = T, \quad \frac{\partial u}{\partial x}(L) = 0$$

General steady-state sol.

$$\cancel{\frac{d^2u}{dx^2}} \quad \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q = 0$$

Because thermal properties of rod are constant $\Rightarrow$ $K_0 = \text{const.}$

$$K_0 \frac{d^2 u}{dx^2} + x^2 K_0 = 0$$

$$\frac{d^2 u}{dx^2} + x^2 = 0$$

General sol of nonhomogeneous problem is obtained by twice integration over x:

$$\boxed{u = -\frac{x^4}{12} + C_1 + C_2 x}$$

To satisfy BC:

$$u(0) = C_1 = T$$

$$\frac{\partial u}{\partial x}(L) = -\frac{x^3}{3} + C_2 \Big|_{x=L} = -\frac{L^3}{3} + C_2 = 0$$

$$\Rightarrow C_2 = \frac{L^3}{3}$$

$$\Rightarrow \boxed{u = -\frac{x^4}{12} + \frac{L^3}{3}x + T}$$

1.4.3

$u(0)=0$ 

$u(2)=0$

$c\rho = 1$, $K_0 = 1$, $Q = 1$ $\qquad$ $c\rho = 2$, $K_0 = 2$, $Q = 0$

perfect thermal contact.

According to 1.3.?, it implies

$$(1)\ \begin{cases} u(1_+,t) = u(1_-,t) \\ \underbrace{K_0(1_-)}_{=1} \dfrac{\partial u}{\partial x}(1_-,t) = \underbrace{K_0(1_+)}_{=2} \dfrac{\partial u}{\partial x}(1_+,t) \end{cases}$$

$$\Rightarrow \frac{\partial u}{\partial x}(1_-,t) = 2\frac{\partial u}{\partial x}(1_+,t)$$

In rod 1:

$$1 \cdot \frac{d^2u}{dx^2} + 1 = 0 \Rightarrow u = -\frac{x^2}{2} + c_1 + c_2 x$$

$$u(0) = c_1 = 0$$

$$(2)\ \begin{cases} \dfrac{\partial u}{\partial x}(1_-,t) = -x + c_2\Big|_{x=1} = -1 + c_2 \\ u(1_-,t) = -\dfrac{1}{2} + c_2 \end{cases}$$

1.4.10

$u_t = u_{xx} + 4 \quad \Rightarrow \quad K = 1$
$\qquad\qquad\qquad\qquad\qquad c\rho = 1$

$u(x,0) = f(x)$

$\frac{\partial u}{\partial x}(0,t) = 5$

$\frac{\partial u}{\partial x}(L,t) = 6$

Find total thermal energy $E(t)$ in a rod.

$$E = \int_0^L u(x,t)\, dx$$

Using eq (1.2.7):

$$\frac{dE}{dt} = \int_0^L \frac{\partial u}{\partial t}\, dx = \int_0^L \left(-\frac{\partial \phi}{\partial x} + \underset{\overset{\parallel}{4}}{Q}\right) dx$$

$$= \phi(0) - \phi(L) + 4L$$

But $\phi(x) = -\frac{\partial u}{\partial x}$

$\Rightarrow \phi(0) = -5$

$\quad\ \ \phi(L) = -6$

$$\Rightarrow \frac{dE}{dt} = -5 + 6 + 5L = 4L + 1, \quad \text{also } E(0) = \int_0^L f(x)\,dx$$

$$\Rightarrow \boxed{E(t) = \int_0^L f(x)\,dx + (4L+1)t}$$

2.3.1a

$$u_t = \frac{\kappa}{r}\partial_r\left(r\partial_r u\right)$$

$$u = \phi(r)G(t)$$

$$\Rightarrow \phi\frac{dG}{dt} = \frac{\kappa G}{r}\partial_r\left(r\partial_r\phi\right)$$

$$\Rightarrow \frac{\frac{dG}{dt}}{\kappa G} = \frac{\cancel{\kappa}1}{r\phi}\partial_r\left(r\partial_r\phi\right) = -\lambda$$

$$\Rightarrow \boxed{\begin{cases} \frac{dG}{dt} = -\kappa\lambda G \\ \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda\phi \end{cases}}$$

2.3.1c

$$u_{xx} + u_{yy} = 0$$

$$u = \phi(x)h(y)$$

$$\Rightarrow h\phi_{xx} + \phi h_{yy} = 0$$

dividing by ϕh:

$$\frac{\phi_{xx}}{\phi} = -\frac{h_{yy}}{h} = -\lambda \Rightarrow \boxed{\begin{array}{l} \frac{d^2\phi}{dx^2} = -\lambda\phi \\ \frac{d^2h}{dy^2} = \lambda h \end{array}}$$

2.3.2.b

$$(*) \quad \frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

$$\phi(0) = 0, \quad \phi(1) = 0$$

General sol of $(*)$:

$\underline{\lambda > 0}$

$$\phi = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

$$\phi(0) = c_1 = 0$$

$$\phi(1) = c_2 \sin(\sqrt{\lambda}) = 0$$

$$\Rightarrow \sqrt{\lambda} = \pi n, \quad \lambda = n^2\pi^2, \quad n = 1, 2, \ldots$$

$\underline{\lambda = 0}$

$$\phi = c_1 + c_2 x$$

$$\phi(0) = c_1 = 0$$

$$\phi(1) = c_2 = 0 \quad \Rightarrow \text{trivial sol.}$$

$\underline{\lambda < 0}$

$\lambda < 0$

$\phi = c_1 e^{\sqrt{-\lambda} x} + c_2 e^{-\sqrt{-\lambda} x}$

$\phi(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$\phi(1) = c_1 \underbrace{\left(e^{\sqrt{-\lambda}\, 1} - e^{-\sqrt{-\lambda}\, 1}\right)}_{>0} \Rightarrow c_1 = 0$

$\Rightarrow \phi \equiv 0$ - trivial sol only.

I.e. only solutions

for $\lambda = n^2 \pi^2$, $n = 1, 2, \ldots$

2.3.2.d

$\phi(0) = 0$, $\frac{d\phi}{dx}(L) = 0$

$\lambda > 0$ $\phi = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$

$\phi(0) = 0 \Rightarrow c_1$

$$\frac{d\,\varphi(L)}{dx} = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,x)\Big|_{x=L} = 0$$

$$\Rightarrow \sqrt{\lambda}\,L = n\pi - \frac{\pi}{2}, \quad n = 1, 2, \ldots$$

$$\lambda = \frac{\left(n\pi - \frac{\pi}{2}\right)^2}{L^2}, \quad n = 1, 2, \ldots$$

$\underline{\lambda = 0}$

$$\Rightarrow \Phi = c_1 + c_2 x$$

$$\Phi(0) = c_1 = 0$$

$$\frac{d\,\Phi(L)}{dL} = c_2 = 0 \Rightarrow \text{trivial sol.}$$

$\lambda = 0$ is <u>Not</u> eigenvalue

$\underline{\lambda < 0}$

$$\Phi = c_1 e^{\sqrt{-\lambda}\,x} + c_2 e^{-\sqrt{-\lambda}\,x}$$

$$\Phi(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$\frac{d\Phi(L)}{dL} = \sqrt{-\lambda}\left(c_1 e^{\sqrt{-\lambda}\,L} + c_1 e^{-\sqrt{-\lambda}\,L}\right)$$

$$= c_1\sqrt{-\lambda}\,\underbrace{\left(e^{\sqrt{-\lambda}\,L} + e^{-\sqrt{-\lambda}\,L}\right)}_{\neq 0} = 0 \Rightarrow c_1 = 0$$

$\Rightarrow$ only trivial solution $c_1 = c_2 = 0$

$\Rightarrow \lambda < 0$ - <u>Not the eigenvalue</u>

HW02 Solutions

2.3.3 b

$$u_t = k\, u_{xx}$$
$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = 3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)$$

From (2.3.30) of textbook:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

IC: $\quad u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = 3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)$

$\Rightarrow$ by visual inspection $B_1 = 3$, $B_3 = -1$,
$B_2 = B_4 = B_5 = \dots 0$

Or by (2.3.32):

$$B_n = \frac{2}{L}\int_0^L \left(3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

~~by eqns from Wiki~~

$$\cancel{= \frac{2}{L}\int_0^L \frac{3}{2}\left[\cos\left(\frac{(1-n)\pi x}{L}\right) - \cos\left(\frac{(1+n)\pi x}{L}\right)\right]dx}$$

$$= \frac{2}{L}\cdot 3\cdot\frac{L}{2}\,\delta_{n,1} + \frac{2}{L}(-1)\frac{L}{2}\,\delta_{n,3}$$

$$= 3\delta_{n,1} - 1\,\delta_{n,3}\,, \quad \text{i.e. } B_1 = 3,\ B_3 = -1,\ B_2 = B_4 = \dots 0$$

$$\boxed{u(x,t) = 3\sin\left(\frac{\pi x}{L}\right)e^{-k\left(\frac{\pi}{L}\right)^2 t} - \sin\left(\frac{3\pi x}{L}\right)e^{-9k\left(\frac{\pi}{L}\right)^2 t}}$$

2.3.3. c $\quad u(x,0) = 2\cos\frac{3\pi x}{L}$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n = \frac{2}{L}\int_0^L 2\cos\left(\frac{3\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \left\{ \text{by } \cos\theta\cdot\sin\varphi = \frac{1}{2}\left[\sin(\theta+\varphi) - \sin(\theta-\varphi)\right] \text{ from Wiki} \right\}$$

$$= \frac{2}{L}\,\frac{2}{2}\int_0^L \left(\sin\frac{(3+n)\pi x}{L} - \sin\frac{(3-n)\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left[-\frac{L}{\pi(3+n)}\cos\left(\frac{(3+n)\pi x}{L}\right)\cdot\left(-\underbrace{\delta_{n+3,0}}_{0 \text{ because } n\geq 1}+1\right) + \frac{L}{\pi(3-n)}\cos\left(\frac{\pi x}{L}\right)\left(-\delta_{3-n,0}+1\right)\right]\Bigg|_{x=0}^{L}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^{3+n}-1}{3+n}(-1) + \frac{1}{3-n}\left[(-1)^{3-n}-1\right]\left(1-\delta_{3-n,0}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n+1}{3+n} + \frac{(-1)^{n+1}}{n-3}\left(1-\delta_{3-n,0}\right)\right]$$

2.3.4 a

$$u_t = k\frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = u(L,t) = 0, \quad u(x,0) = f(x)$$

total heat energy $= \int_0^L c\rho u A \, dx$

$$= c\rho A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \int_0^L \sin\frac{n\pi x}{L}\, dx$$

$$= c\rho A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \frac{-\cos(n\pi)+1}{\frac{n\pi}{L}}$$

$$= c\rho A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \frac{1-(-1)^n}{\frac{n\pi}{L}},$$

where $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\, dx$

according to (2.3.35)

2.3.4. b

Heat flux to the right $= -K_0 \dfrac{\partial u}{\partial x}$

Total heat flow to the right $= -K_0 \dfrac{\partial u}{\partial x} A$

At $x=0$: heat flow out $= K_0 \left.\dfrac{\partial u}{\partial x}\right|_{x=0} A$

$$= K_0 A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \frac{n\pi}{L} \left.\underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{=1}\right|_{x=0}$$

$$= K_0 A \sum_{n=1} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \frac{n\pi}{L}$$

At $x=L$ heat flow out $= -K_0 \left.\dfrac{\partial u}{\partial x}\right|_{x=L} A$

$$= -K_0 A \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \frac{n\pi}{L} \underbrace{\cos(n\pi)}_{=(-1)^n}.$$

2.3.4.c From conservation of thermal energy

$$\underbrace{\frac{d}{dt}\int_0^L u\,dx}_{\substack{\text{rate of change}\\ \text{of thermal energy}}} = k\frac{\partial u}{\partial x}\Big|_0^L = \underbrace{k\frac{\partial u}{\partial x}(L) - k\frac{\partial u}{\partial x}(0)}_{\text{fluxes through ends}}$$

$$k = \frac{K_0}{c\rho}$$

2.3.5

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_0^L\left[\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right]dx$$

$$= \frac{1}{2}\Bigg[\delta_{n,m}L + (1-\delta_{n,m})\frac{\sin\frac{(n-m)\pi x}{L}}{\frac{(n-m)\pi}{L}}\Bigg|_0^L$$

$$- \underbrace{\delta_{n,-m}}_{=0}L + (1-\underbrace{\delta_{n,-m}}_{=0})\frac{\sin\frac{(n+m)\pi x}{L}}{\frac{n-m\,\pi}{L}}\Bigg|_0^L\Bigg]$$

because $n, m > 0$

$$= \frac{L}{2}\delta_{n,m}$$

2.3.8

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} - \alpha u \quad , \alpha > 0 , \kappa > 0$$

$$u(0,t) = u(L,t) = 0.$$

(a) Equilibrium sol:

$$\kappa \frac{d^2 u}{dx^2} = \alpha u$$

General ODE solution:

$$u(x) = a\cosh\left(\sqrt{\tfrac{\alpha}{\kappa}}x\right) + b\sinh\left(\sqrt{\tfrac{\alpha}{\kappa}}\, x\right)$$

$$u(0) = a = 0$$
$$u(L) = b\sinh\left(\sqrt{\tfrac{\alpha}{\kappa}}\, L\right) = 0$$
$$\Rightarrow b = 0$$

$\Rightarrow$ $\boxed{u(x) \equiv 0}$ - only equilibrium.

(b) Separation of variables:

$$u = \phi(x) \cdot h(t)$$

$$\Rightarrow \phi \frac{dh}{dt} + \alpha \phi h = \kappa h \frac{d^2\phi}{dx^2}$$

$$\frac{\frac{dh}{dt}}{\kappa h} + \frac{\alpha}{\kappa} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$

ODE BVP:

$$\begin{cases} \dfrac{d^2\Phi}{dx^2} = -\lambda \Phi \\ \Phi(0) = \Phi(L) = 0 \end{cases} \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \; n = 1, 2, \ldots$$

Eigenvalues:

Eigenfunctions:

$$\Phi = \sin\left(\frac{n\pi x}{L}\right)$$

For h:

$$\frac{dh}{dt} + \alpha h = -k\lambda h = -k\left(\frac{n\pi}{L}\right)^2 h$$

$$\Rightarrow h = c\, e^{-\alpha t} \cdot e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Superposition:

$$\boxed{u(x,t) = e^{-\alpha t} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

$$IC: \; u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

orthogonality $\Rightarrow \boxed{b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)}$.

As $t \to \infty \Rightarrow u \to 0$ as in equilibrium in (a).

2.4.1 a $\quad u_t = k u_{xx}, \quad 0 < x < L, \quad t > 0.$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, \quad t > 0$$

In text (2.4.19) u is found as:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

with (2.4.23-24)

$$\begin{cases} A_0 = \frac{1}{L} \int_0^L f(x)\,dx \\ A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \end{cases}$$

$\Rightarrow$ (a) $\quad u(x,0) = \begin{cases} 0, & x < \frac{L}{2} \\ 1, & x > \frac{L}{2} \end{cases}$

$$A_0 = \frac{1}{L} \int_{\frac{L}{2}}^{L} dx = \frac{1}{2}$$

$$A_n = \frac{2}{L} \int_{L/2}^{L} \cos \frac{n\pi x}{L}\, dx = \frac{2}{L} \cdot \frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)\Big|_{x=\frac{L}{2}}^{L}$$

$$= \frac{2}{n\pi}\left(\underbrace{\sin(n\pi)}_{=0} - \sin\frac{n\pi}{2}\right) = -\frac{2}{n\pi} \begin{cases} 0, & n = 2,4,6,\ldots \\ 1, & n = 1,5,9,\ldots \\ -1, & n = 3,7,11,\ldots \end{cases}$$

2.4.1 (b) $u(x,0) = 6 + 4\cos\frac{3\pi x}{L}$

by inspection: $A_0 = 6$, $A_3 = 4$,

others $= 0$.

2.4.1 (c) $u = -2\sin\left(\frac{\pi x}{L}\right)$

$$A_0 = \frac{1}{L}\int_0^L -2\sin\frac{\pi x}{L}\,dx = \frac{2}{\pi}\cos\frac{\pi x}{L}\Big|_0^L = \frac{2}{\pi}(-1+\cos\pi)$$

$$= -\frac{4}{\pi}$$

$$A_n = -\frac{4}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx$$

$\{$ use $\sin\theta\cos\varphi = \frac{1}{2}\left[\sin(\theta+\varphi) + \sin(\theta-\varphi)\right]\}$

$$= -\frac{2}{L}\int_0^L \left[\sin\left(\frac{(n+1)\pi x}{L}\right) + \sin\left(\frac{(n-1)\pi x}{L}\right)\right]dx$$

$$= \frac{2}{L}\frac{L}{(n+1)\pi}\cos\left(\frac{(n+1)\pi x}{L}\right)\Big|_0^L - \frac{2}{L}\frac{L}{(n-1)\pi}\cos\frac{(n-1)\pi x}{L}\Big|_0^L \cdot \delta_{n,1}$$

$$= \frac{2}{(n+1)\pi}\left[(-1)^{n+1}-1\right] - \frac{2}{(n-1)\pi}\left[(-1)^{n-1}-1\right]\delta_{n,1}$$

$$= \begin{cases} -\dfrac{4}{(n+1)\pi} - \dfrac{4}{(n-1)\pi}\delta_{n,1}, & n\text{-even} \\ 0, & n \text{ odd}, \end{cases}$$

2.4.2

$$\boxed{\begin{aligned} & u_t = k u_{xx}, \\ & \frac{\partial u}{\partial x}(0,t) = 0, \quad u(L,t) = 0 \\ & u(x,0) = f(x) \end{aligned}}$$

$$u = \phi(x)\, h(t)$$

$$\phi \cdot \frac{dh}{dt} = k \cdot h \frac{d^2\phi}{dx^2}$$

$$\Rightarrow \quad \frac{\frac{dh}{dt}}{h} = k \, \frac{\frac{d^2\phi}{dx^2}}{\phi} = -\lambda$$

$$\frac{dh}{dt} = -\lambda h \quad \Rightarrow \quad h = c e^{-\lambda t}$$

$$\frac{d^2\phi}{dx^2} = -\frac{\lambda}{k}\phi \quad \Rightarrow \quad \phi = A \cos\left(\sqrt{\frac{\lambda}{k}}\, x\right) + B \sin\left(\sqrt{\frac{\lambda}{k}}\, x\right)$$

$$\Phi'(0) = 0 \Rightarrow \left. -A\sqrt{\tfrac{\lambda}{k}}\sin\left(\sqrt{\tfrac{\lambda}{k}}x\right) + B\sqrt{\tfrac{\lambda}{k}}\cos\left(\sqrt{\tfrac{\lambda}{k}}x\right)\right|_{x=0}$$

$$= B\sqrt{\tfrac{\lambda}{k}} = 0 \Rightarrow B = 0$$

$$\Phi(L) = A\cos\left(\sqrt{\tfrac{\lambda}{k}}L\right) = 0$$

$$\Rightarrow \sqrt{\tfrac{\lambda}{k}}L = \left(n+\tfrac{1}{2}\right)\pi \quad , n = 0, 1, \ldots$$

$$\Rightarrow \lambda = \frac{\left(n+\frac{1}{2}\right)^2\pi^2}{L^2}k \quad - \text{eigenvalues}$$

Eigenfunctions: $\cos\left(\frac{\left(n+\frac{1}{2}\right)\pi}{L}x\right)$

Superposition:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left[\frac{\left(n+\frac{1}{2}\right)\pi x}{L}\right] e^{-\frac{\left(n+\frac{1}{2}\right)^2\pi^2}{L^2}kt}$$

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{\left(n+\frac{1}{2}\right)\pi x}{L}\right) = f(x)$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{\left(n+\frac{1}{2}\right)\pi x}{L}\,dx \quad , n = 0, 1, \ldots$$

2.4.3

$$\phi_{xx} = -\lambda \phi$$

$$\phi(0) = \phi(2\pi), \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi)$$

Set $y = x - \pi$

$$\Rightarrow \begin{cases} \phi_{yy} = -\lambda \phi \\ \phi(-\pi) = \phi(\pi) \\ \dfrac{d\phi(-\pi)}{dy} = \dfrac{d\phi(\pi)}{dy} \end{cases}$$

$L = \pi$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 = n^2, \quad n = 0, 1, \ldots$$

Eigenfunctions

$$\cos\left(\frac{n\pi y}{L}\right) = \cos(ny)$$

$$= \cos(n(x-\pi)) = (-1)^n \cos(nx)$$

$$\sin\left(\frac{n\pi y}{L}\right) = \sin(ny)$$

$$= \sin(n(x-\pi))$$

$$= (-1)^n \sin nx$$

$(-1)^n$ can be included into arbitrary constant $\Rightarrow$ eigenfunctions $\sin(nx)$, $\cos(nx)$

2.5.1a

$$\Delta u = 0$$

$$0 \le x \le L, \quad 0 \le y \le H$$

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(L, y) = 0,$$

$$u(x, 0) = 0, \quad u(x, H) = f(x)$$

$$u = h(x)\,\phi(y)$$

$$\Rightarrow \quad \frac{\frac{d^2 h}{dx^2}}{h} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} = -\lambda$$

$$\Rightarrow \begin{cases} \frac{d^2 h}{dx^2} = -\lambda h \\ h'(0) = 0 \\ h'(L) = 0 \end{cases} \Rightarrow \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \; n = 0, 1, 2, \ldots$$

$$h = \cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad \frac{d^2\phi}{dy^2} = \left(\frac{n\pi}{L}\right)^2 \phi$$

$\underline{n = 0}$: $\phi = C_1 + C_2 y$, $\phi(0) = 0 \Rightarrow C_1 = 0$

$\underline{n \neq 0}$: $\Phi = c_1 \cosh\left(\frac{n\pi y}{L}\right) + c_2 \sinh\left(\frac{n\pi y}{L}\right)$

$$\Phi(0) = c_1 = 0$$

Superposition:

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$$

the nonhomogeneous BC:

$$f(x) = u(x,H) = A_0 H + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi H}{L}\right) \cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow A_0 H = \frac{1}{L} \int_0^L f(x)\, dx$$

$$A_n \sinh\left(\frac{n\pi H}{L}\right) = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

2.5.1.e

$$u(0,y) = u(L,y) = 0$$

$$u(x,0) = \frac{\partial u}{\partial y}(x,0)$$

$$u(x,H) = f(x)$$

$$u = \phi(x)\, h(y)$$

$$\frac{\frac{d^2\phi(x)}{dx^2}}{\phi} = -\frac{\frac{d^2h}{dy^2}}{h} = -\lambda$$

ODE BVP :

$$\begin{cases} \frac{d^2\phi}{dx^2} = -\lambda\phi \\ \phi(0) = \phi(L) = 0 \end{cases} \Rightarrow$$

Eigenvalues

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

Eigenfunctions :

$$\phi = \sin \frac{n\pi x}{L}, \quad n = 1, 2, \ldots$$

$$\begin{cases} \frac{d^2h}{dy^2} = \left(\frac{n\pi}{L}\right)^2 h \\ h(0) = \frac{dh}{dy}(0) \end{cases} \Rightarrow$$

general sol

$$h = c_1 \cosh\left(\frac{n\pi y}{L}\right) + c_2 \sinh\left(\frac{n\pi y}{L}\right)$$

$$h(0) - \frac{dh}{dy}(0) = c_1 - \frac{n\pi}{L}\left[\sinh\left(\frac{n\pi y}{L}\right)\right.$$

$$+ c_2 \cosh\left(\frac{n\pi y}{L}\right)\Bigg]\Bigg|_{y=0}$$

$$= c_1 - \frac{n\pi}{L} c_2 = 0 \Rightarrow c_1 = c_2 \frac{n\pi}{L}$$

$$\Rightarrow h = \cosh\left(\frac{n\pi y}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi y}{L}\right)$$

Superposition

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \left[\cosh\left(\frac{n\pi y}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi y}{L}\right)\right] \times \sin\left(\frac{n\pi x}{L}\right)}$$

$n = 1, 2, \ldots$, where A_n is determined from

$$\text{BC} \quad f(x) = \sum_{n=1}^{\infty} A_n\left[\cosh\left(\frac{n\pi H}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi H}{L}\right)\right] \times \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \boxed{A_n\left[\cosh\left(\frac{n\pi H}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi H}{L}\right)\right] = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

(2.5.2a)

$0 < x < L$

$0 < y < H$

$\frac{\partial u}{\partial x}(0,y)=0$, $\frac{\partial u}{\partial y}(x,0)=0$

$\frac{\partial u}{\partial x}(L,y)=0$, $\frac{\partial u}{\partial y}(x,H)=f(x)$

(a) the total heat flow across the boundary must be equal to zero in equilibrium (i.e. without sources in Laplace eq)

$\Rightarrow \int_0^L f(x)\,dx = 0$

(b) using (2.5.16) : $u = h(x)\,\phi(y)$

$$\frac{h_{xx}}{h} = -\frac{\phi_{yy}}{\phi} = -\lambda$$

$$\begin{cases} h_{xx} = -\lambda h \\ \frac{\partial h}{\partial x}(0) = \frac{\partial h}{\partial x}(L) = 0 \end{cases}$$ homogeneous BVP

~~$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \ldots$~~

~~$h(x) = \cos\frac{n\pi x}{L}$ eigenfunctions~~

$$h = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

$$h_x = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda}\, x) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda}\, x)$$

$$h_x(0) = c_2 \sqrt{\lambda} = 0 \Rightarrow c_2 = 0$$

$$h_x(L) = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} L) = 0$$

$$\Rightarrow \lambda = 0 \text{ or } \sqrt{\lambda} L = n\pi, \; n = 0, 1 \ldots$$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2 \ldots$$

Eigenfunctions: $\cos\left(\frac{n\pi}{L} x\right)$

$\lambda = 0$: $h = c_1 + c_2 x$

$h_x = c_2 = 0$

$$\begin{cases} \dfrac{d^2\Phi}{dy^2} = \left(\dfrac{n\pi}{L}\right)^2 \Phi \\ \dfrac{d\Phi}{dy}(0) = 0 \end{cases}$$

$$\Rightarrow \Phi = \cancel{\sinh}\ \cosh\left(\frac{n\pi}{L} y\right)$$

Superposition:

$$u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right)$$

$$\frac{\partial u(x,H)}{\partial y} = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi H}{L}\right) = f(x)$$

$$\Rightarrow A_n \cosh\left(\frac{n\pi H}{L}\right) = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$n = 1, 2, \ldots$

$$\boxed{u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right),}$$

(*) A_0 – arbitrary.

(c (optional)) : to find A_0

$$\frac{\partial u}{\partial t} = k \nabla^2 u, \quad u(x,y,0) = g(x,y)$$

Initial heat energy: $E(0) = c\rho \int_0^L \int_0^H u(x,y,0)\, dx\, dy$

$$= c\rho \int_0^L \int_0^H g(x,y)\, dx\, dy$$

Rate of change of heat energy:

$$\frac{dE}{dt} = -\int_0^L K_0 \frac{\partial u}{\partial y}(x,H)\, dx = -K_0 \int_0^L f(x) dx$$

$= 0$ according to (a)

$$\Rightarrow E(t)\Big|_{t\to\infty} = E(0) = c\rho \int_0^L \int_0^H g(x,y)\,dx\,dy$$

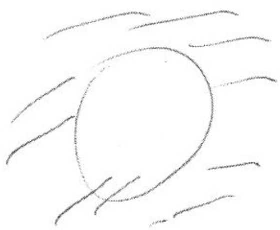
$$\text{But } E(\infty) = c\rho \int_0^L \int_0^H u(x,y)\,dx\,dy$$

(using solution (*) of (8))

$$= c\rho \cdot L \cdot H \cdot A_0 + 0$$

$$\Rightarrow A_0 = \frac{\int_0^L \int_0^H u(x,y)\,dx\,dy}{LH}$$

2.5.3 Outside of a circular disk $r \geq a$



$$\nabla^2 u = \frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_\theta^2 u = 0$$

$$u = \phi(\theta)\,G(r)$$

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda$$

$$\begin{cases} \dfrac{d^2\Phi}{d\theta^2} = -\lambda\Phi \\ \Phi(-\pi) = \Phi(\pi) \\ \dfrac{d\Phi}{d\theta}(-\pi) = \dfrac{d\Phi}{d\theta}(\pi) \end{cases}, \quad l = \pi$$

$\Rightarrow \lambda = \left(\frac{n\pi}{l}\right)^2 = n^2$ - eigenvalues

$\sin(n\theta)$, $\cos(n\theta)$ - eigenfunctions.

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = n^2$$

$$r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2G = 0$$

$n \neq 0$: $G = c_1 r^n + c_2 r^{-n}$

$n = 0 \Rightarrow G = \bar{c}_1 + \bar{c}_2 \ln r$

to have $|u| < \infty$ as $r \to \infty$ $\Rightarrow c_1 = \bar{c}_2 = 0$

Superposition:

$$u = \sum_{n=0}^{\infty} A_n r^{-n}\cos(n\theta) + \sum_{n=1}^{\infty} B_n r^{-n}\sin(n\theta)$$

(a) $u(a,\theta) = \ln 2 + 4\cos(3\theta)$

$\Rightarrow A_0 = \ln 2$, $A_3 a^{-3} = 0$, $A_n = B_n = 0$ for other n

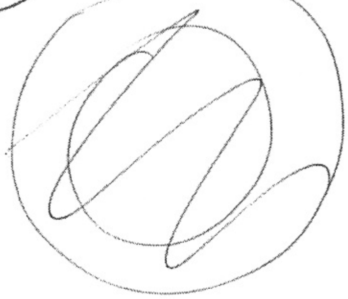
(b) $u(a,\theta) = f(\theta)$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$A_n a^{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$B_n a^{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta$$

2.5.6.a



$0 < \theta < \pi$

$0 < r < a$

$$\nabla^2 u = 0$$

$u = 0$ at diameter and $u(a,\theta) = g(\theta)$

Similar to (2.5.37) but different BC:

$$\begin{cases} \dfrac{d^2\Phi}{d\theta^2} = -\lambda\Phi \\ \Phi(0) = \Phi(\pi) = 0 \end{cases} \Rightarrow \lambda = \left(\frac{n\pi}{\pi}\right)^2 = n^2 \text{ eigenvalues}$$

$\sin(n\theta)$, $n = 1, 2, 3 \ldots$ – eigenfunctions.

$G = c_1 r^n + c_2 r^{-n}$, $c_2 = 0$ to have $|u| < \infty$ at $r = 0$ (7)

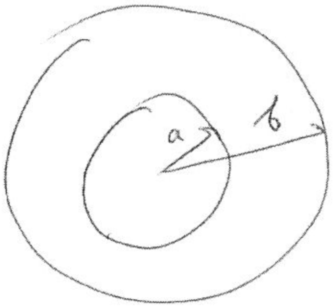
Superposition:

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin(n\theta)}$$

BC at $r = a$: $g(\theta) = \sum_{n=1}^{\infty} A_n a^n \sin(n\theta)$

$$\Rightarrow \boxed{A_n a^n = \frac{2}{\pi} \int_0^{\pi} g(\theta) \sin(n\theta)\, d\theta}$$

2.5.8 a



$a < r < b$

$\nabla^2 u = 0$

BC: $u(a,\theta) = f(\theta)$
$u(b,\theta) = g(\theta)$

$u = \Phi(\theta)\, G(r) \Rightarrow$

$$\nabla^2 u = \frac{1}{r} \partial_r (r \partial_r u) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{r}{G} \frac{d}{dr}\left(r \frac{dG}{dr}\right) = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\theta^2} = \lambda \qquad -\pi < \theta < \pi$$

Periodicity in φ: $\begin{cases} \varphi(\pi) = \varphi(-\pi) \\ \dfrac{d\varphi}{d\theta}(\pi) = \dfrac{d\varphi}{d\theta}(-\pi) \\ \dfrac{d^2\varphi}{d\theta^2} = -\lambda\varphi \end{cases}$

$$\Rightarrow \varphi = c_1 \cos(\sqrt{\lambda}\,\theta) + c_2 \sin(\sqrt{\lambda}\,\theta)$$

$$\frac{d\varphi}{d\theta} = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,\theta) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,\theta)$$

<u>BC:</u> $\cancel{c_1 \cos(\sqrt{\lambda}\,\pi)} + c_2 \sin(\sqrt{\lambda}\,\pi) = \cancel{c_1 \cos(\sqrt{\lambda}\,\pi)} - c_2 \sin\sqrt{\lambda}\,\pi$

$$-c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi) + \cancel{c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi)} = c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi) + \cancel{c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi)}$$

$$\Rightarrow \sin(\sqrt{\lambda}\,\pi) = 0 \Rightarrow \sqrt{\lambda}\,\pi = n\pi, \quad n \in \mathbb{Z}$$

$\lambda = n^2$, $n = 0, 1, \ldots$ - eigenvalues

$\cancel{\lambda \neq 0}$ $\cos(n\theta)$, $\sin(n\theta)$ - eigenfunctions

$\cancel{\lambda = 0 \Rightarrow \bar{c}_1 + \bar{c}_2 \ln r}$

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = n^2 \Rightarrow r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2 G = 0$$

$\underline{\lambda \neq 0}$ $\quad G = c_1 r^n + c_2 r^{-n}$, $\quad n = 1, 2, \ldots$

$\underline{\lambda = 0}$ $\quad G = \bar{c}_1 + \bar{c}_2 \ln r$

It is convenient to choose instead $r^{\pm n}$ the following eigenfunctions:

$$G_1(r) = \begin{cases} \ln \frac{r}{a}, & n = 0 \\ \left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n, & n \neq 0 \end{cases} \Rightarrow G_1(a) = 0$$

$$G_2(r) = \begin{cases} \ln \left(\frac{r}{b}\right), & n = 0 \\ \left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^n, & n \neq 0 \end{cases} \Rightarrow G_2(b) = 0$$

Superposition:

$$u(r, \theta) = \sum_{n=0}^{\infty} \cos(n\theta) \left[A_n \cancel{\Phi_1} G_1(r) + B_n G_2(r) \right]$$

$$+ \sum_{n=1}^{\infty} \sin(n\theta) \left[C_n G_1(r) + D_n \Phi_2(r) \right]$$

$\underline{BC}$: $r = a$: $\quad f(\theta) = \sum_{n=0}^{\infty} \cos(n\theta) \left[B_n \cancel{\Phi_2} G_2(a) \right]$

$$+ \sum_{n=1}^{\infty} \sin(n\theta) \left[D_n \cancel{\Phi_2} G_2(a) \right]$$

$\underline{r = b}$ $\quad g(\theta) = \sum_{n=0}^{\infty} \cos(n\theta) \left[A_n G_1(b) \right]$

$$+ \sum_{n=1}^{\infty} \sin(n\theta) \left[C_n G_1(b) \right]$$

$\Rightarrow$ By orthogonality

$$B_0\, G_2(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta.$$

$$B_n\, G_2(a) = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(n\theta)\, f(\theta)\,d\theta \quad , n = 1, 2, \ldots$$

$$A_0\, G_1(b) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,d\theta$$

$$A_n\, G_1(b) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\, \cos(n\theta)\,d\theta \quad , n = 1, 2, \ldots$$

$$C_0\, G_1(b) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,d\theta$$

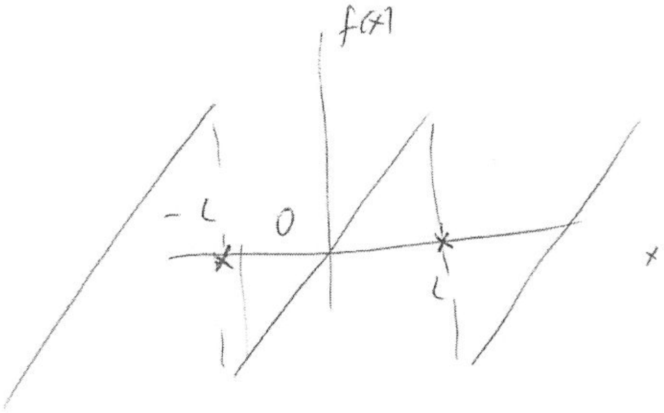
$$C_n\, G_1(b) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\theta)\, g(\theta)\,d\theta$$

$$D_0\, G_2(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$D_n\, G_2(a) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\theta)\, f(\theta)\,d\theta$$

3.2.2.a

$-L \le x \le L$



$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\frac{x^2}{2}\Big|_{-L}^{L} = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx = 0 \Big]$$ because x is odd!

$$b_n = \frac{2L}{\pi n}(-1)^{n+1} \quad \text{(from class)}$$

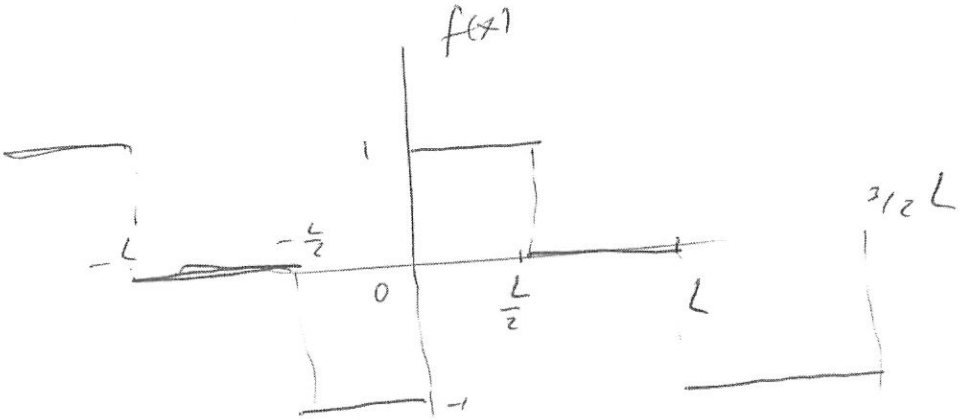
$$\Rightarrow \quad x \sim \sum_{n=1}^{\infty} \frac{2L}{\pi n}(-1)^{n+1}\sin\left(\frac{n\pi}{L}x\right)$$

3.3.2d

(12)

$$f(x) = \begin{cases} 1 & , x < \frac{L}{2} \\ 0 & , x > \frac{L}{2} \end{cases}$$

· Odd extension



$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n}{L} x\right) dx$$

$$= \frac{2}{L} \int_0^{L/2} f(x) \sin\left(\frac{\pi n x}{L}\right) dx = \frac{2}{L} \left(-\frac{L}{\pi n}\right) \cos\left(\frac{n\pi}{L} x\right)\Big|_0^{L/2}$$

$$= \frac{2}{n\pi}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)$$

$$\cancel{= \frac{2}{L}\sin\left(\frac{n\pi x_0}{L}\right)(\alpha-\beta) - \frac{n\pi}{L}a_n}$$

3.4.9

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x,t)$$

$$u(0,t)=0$$
$$u(L,t)=0$$

$q(x,t)$ - piecewise smooth in x $\forall t$.

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L} \qquad (*)$$

Because $\frac{\partial^2 u}{\partial x^2}$ is piecewise smooth and $u(0,t)=0=u(L,t)$

$\Rightarrow$ and $u(x,t)$ is continuous

$\Rightarrow$ we can differentiate $(*)$ term by term:

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right) b_n \cos\left(\frac{n\pi x}{L}\right) \qquad (**)$$

Because $\frac{\partial^2 u}{\partial x^2}$ is piecewise smooth

$\Rightarrow$ we can differentiate $(**)$ term by term:

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 b_n \sin\frac{n\pi x}{L}$$

thus from p. 119 and from class:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{d b_n}{dt}\sin\frac{n\pi x}{L} \sim -k\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 b_n \sin\frac{n\pi x}{L} + q$$

$$\Rightarrow \boxed{\frac{d b_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n = \frac{2}{L}\int_0^L q(x,t)\sin\left(\frac{n\pi x}{L}\right)dx}$$

3.4.12

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + e^{-t} + e^{-2t}\cos\frac{3\pi x}{L}$$

(assume $2 \neq k\left(\frac{3\pi}{L}\right)^2$)

BC: $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$

IC: $u(x,0) = f(x)$

the eigenfunctions of homogeneous ODE BVP ~~problem~~ are $\cos\frac{n\pi x}{L}$, $n = 0, 1, 2, \ldots$

$\Rightarrow u \sim \sum_{n=0}^{\infty} A_n(t) \cos\left(\frac{n\pi x}{L}\right)$ – it can

be differentiated (if u is continuous)

since it is a cosine series:

$$\frac{\partial u}{\partial x} \sim \sum_{n=0}^{\infty} A_n \left(-\frac{n\pi}{L}\right) \sin \frac{n\pi x}{L}.$$

this can be differentiated again

(if $\frac{\partial u}{\partial x}$ is continuous) only because

$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$ – according to BC

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} \sim -\sum_{n=0}^{\infty} A_n \left(\frac{n\pi}{L}\right)^2 \cos\left(\frac{n\pi x}{L}\right)$$

$\Rightarrow$ {similar to problem 3.4.9}:

$$\frac{\partial u}{\partial t} + k\frac{\partial^2 u}{\partial x^2} = \sum_{n=0}^{\infty} \left[\frac{dA_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 A_n\right] \cos\frac{n\pi x}{L} = \underbrace{e^{-t} + e^{-2t} \cos\left(\frac{3\pi x}{L}\right)}_{\text{cos series in } x}$$

$$(**) \Rightarrow \frac{dA_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 A_n = \begin{cases} e^{-t}, & n=0 \\ e^{-2t}, & n=3 \\ 0, & n \neq 0, n \neq 3 \end{cases}$$

IC gives $A_0(0) = \frac{1}{L} \int_0^L f(x)\,dx$

$A_n(0) = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \; n \neq 0$

ODE (***) is solved as follows:

$\underline{n=0}$: $\frac{dA_0}{dt} = e^{-t} \Rightarrow A_0(t) - A_0(0) = \int_0^t e^{-t'} dt' = 1 - e^{-t}$

$$\boxed{A_0(t) = A_0(0) + 1 - e^{-t}}$$

$n \neq 0, n \neq 3$: $\frac{dA_n}{dt} = -\kappa\left(\frac{n\pi}{L}\right)^2 A_n$

$$\Rightarrow \boxed{A_n = A_n(0)\, e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t}}$$

$\underline{n=3}$ $A_3 = c(t)\, e^{-\kappa\left(\frac{3\pi}{L}\right)^2 t}$

$$\Rightarrow \frac{dc}{dt} e^{-\kappa\left(\frac{3\pi}{L}\right)^2 t} = e^{-2t}$$

$$c = \underset{A_3(0)}{\underbrace{c(0)}} + \int_0^t e^{-2t' + \kappa\left(\frac{3\pi}{L}\right)^2 t'} dt' = A_3(0) + \frac{e^{-2t + \kappa\left(\frac{3\pi}{L}\right)^2 t} - 1}{-2 + \kappa\left(\frac{3\pi}{L}\right)^2}$$

$$\Rightarrow \boxed{A_3 = A_3(0)\, e^{-\kappa\left(\frac{3\pi}{L}\right)^2 t} + \frac{e^{-2t} - e^{-\kappa\left(\frac{3\pi}{L}\right)^2 t}}{-2 + \kappa\left(\frac{3\pi}{L}\right)^2}}$$