(A) Test 1. Solutions

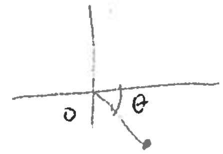
1. Find real and imaginary part of $\frac{(2+i)(-1+i)}{(1-2i)^2}$. $= \frac{(-2-1+i)(1+2i)^2}{(1+4)^2}$

$$= \frac{(-3+i)(1-4+4i)}{25} = \frac{(-3+i)(-3+4i)}{25}$$

$$= \frac{(9-4+i(-3-12))}{25} = \frac{1}{5} - \frac{3i}{5}$$

2. Find all roots of $(-i+\sqrt{3})^{1/2}$. Find expressions for real and imaginary parts of each root.

$$z = -i + \sqrt{3} = r e^{i\theta}$$



$$r = \sqrt{1+3} = 2$$

$$\theta = -\operatorname{Arctan}\frac{1}{\sqrt{3}} = -\frac{\pi}{6} = -\frac{\pi}{2} + \frac{\pi}{3}$$

$$z^{1/2} = \sqrt{2}\, e^{-\frac{i\pi}{2\cdot 6} + \frac{2\pi k}{2}}, \quad k = 0, 1$$

$$c_0 = \sqrt{2}\, e^{-\frac{i\pi}{12}} = \sqrt{2}\cos\frac{\pi}{12} - i\sqrt{2}\sin\frac{\pi}{12}$$

$$c_1 = -\sqrt{2}\, e^{-\frac{i\pi}{12}} = -\sqrt{2}\cos\frac{\pi}{2} + i\sqrt{2}\sin\frac{\pi}{12}$$

Also $$e^{-\frac{i\pi}{2\cdot 6}} = e^{-\frac{i}{2}\left(\frac{\pi}{2} - \frac{\pi}{3}\right)} = e^{-\frac{i\pi}{4}} e^{\frac{i\pi}{6}}$$

$$= \frac{(1-i)}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}} + i\frac{(-\sqrt{3}+1)}{2\sqrt{2}}$$

$$\Rightarrow \operatorname{Re} c_0 = \sqrt{2}\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2}$$

$$\operatorname{Im} c_0 = -\sqrt{2}\sin\frac{\pi}{12} = \frac{-\sqrt{3}+1}{2}$$

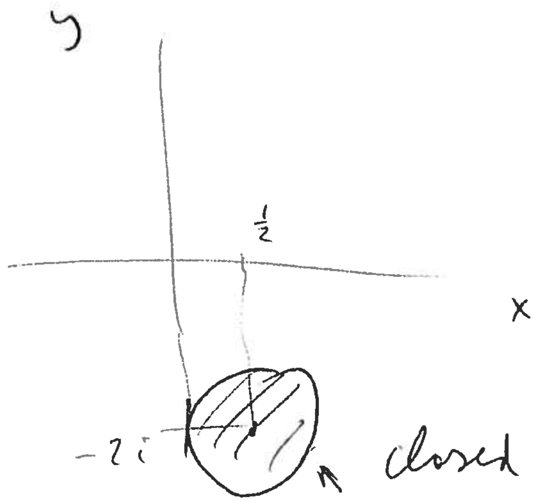
$$\operatorname{Re} c_1 = -\operatorname{Re} c_0 = -\sqrt{2}\cos\frac{\pi}{2} = -\frac{\sqrt{3}+1}{2}$$

$$\operatorname{Im} c_1 = -\operatorname{Im} c_0 = \sqrt{2}\sin\frac{\pi}{12} = \frac{+\sqrt{3}-1}{2}$$

3. (a) Sketch the following set and determine if it is open set, closed set or neither of them: $|2z - 1 + 4i| \leq 1$.
(b) Is this set bounded or unbounded?

$$\left| z - \frac{1}{2} + 2i \right| \leq \frac{1}{2}$$

Disk of radius $\frac{1}{2}$ centered at $z = \frac{1}{2} - 2i$



closed set.
Set is bounded.

4. Determine if limit of $f(z)$ exists for $z \to 0$, where

$$f(z) = \left(\frac{\bar{z}}{z}\right)^3.$$

$z = x + i0, \quad x \neq 0$

$$\left(\frac{x - i0}{x + i0}\right)^3 = 1$$

$z = 0 + iy, \quad y \neq 0$

$$\left(\frac{0 - iy}{0 + iy}\right)^3 = -1$$

$\Rightarrow \lim_{z \to 0} \left(\frac{\bar{z}}{z}\right)^3$ does not exist.

5. Use the definition of derivative to show that
$\frac{d}{dz}\frac{1}{z^3} = -\frac{3}{z^4}$.

$$\lim_{\Delta z \to 0} \frac{\frac{1}{(z+\Delta z)^3} - \frac{1}{z^3}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z^3} - \cancel{z^3} - 3z^2\Delta z - 3z\Delta z^2 - \Delta z^3}{(z+\Delta z)^3 z^3 \Delta z}$$

$$= \lim_{\Delta z \to 0} - \frac{3z^2 - \overbrace{3z\Delta z - \Delta z^2}^{0}}{\underbrace{(z+\Delta z)^3}_{0} z^3} = -\frac{3z^2}{z^6} = -\frac{3}{z^4}$$

6. Show by induction that n-th derivative of $f(z) = 1/z^2$ is given by
$f(z)^{(n)} = (-1)^n(n+1)!z^{-n-2}, n = 1, 2, \dots, \; (z \neq 0)$.
Assume that expression for derivative of z^{-k} is known for any positive integer k.

By induction

① For $n = 1$

$$\left(\frac{1}{z^2}\right)' = -\frac{2}{z^3} = (-1)^n(n+1)!\,z^{-n-2}\Big|_{n=1}$$

② Assume that for $n = m$:

$$\left(\frac{1}{z^2}\right)^{(m)} = (-1)^m(m+1)!\,z^{-m-2}$$

$$\Rightarrow \left(\frac{1}{z^2}\right)^{(m+1)} = \left(\left(\frac{1}{z^2}\right)^{(m)}\right)' = \left((-1)^m(m+1)!\,z^{-m-2}\right)'$$

$$= (-1)^m(m+1)!(-m-2)\,z^{-m-3} = (-1)^{m+1}(m+2)!\,z^{-(m+1)-2}$$

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7. Determine where $f'(z)$ exists and determine its value when:
(a) $f(z) = x^2 + 4 + 2iy$
(b) $f(z) = 2\sqrt[4]{r}e^{i\theta/4}$, $r > 0$, $\alpha < \theta < \alpha + 2\pi$.

(a) $u = x^2 + 4$
$v = 2y$

$u_x = 2x$

$u_y = 0$

$v_x = 0$

$v_y = 2$

$\Rightarrow u_x, u_y, v_x, v_y$ exist and continuous $\forall x, y$

Cauchy - Riemann eqns.

$u_x = v_y \Rightarrow 2x = 2$
$\Rightarrow x = 1$

$u_y = -v_x = 0$

$\Rightarrow f'(z)$ exists along line $x = 1$

$f'(z) = u_x + i v_x = 2x\big|_{x=1} = 2$ $\boxed{\text{for } z = 1 + iy}$

(b) $f(z) = u + iv$ $\quad (r > 0, \ \alpha < \theta < \alpha + 2\pi)$

$u = 2\sqrt[4]{r}\cos\frac{\theta}{4}$

$v = 2\sqrt[4]{r}\sin\frac{\theta}{4}$ 8

$$\begin{cases} u_r = \frac{2}{4}\frac{1}{r^{3/4}}\cos\frac{\theta}{4}, & v_r = \frac{2}{4}\frac{1}{r^{3/4}}\sin\frac{\theta}{4} \\ u_\theta = -\frac{2}{4}\sqrt[4]{r}\sin\frac{\theta}{4}, & v_\theta = \frac{2}{4}\sqrt[4]{r}\cos\frac{\theta}{4} \end{cases}$$

$\left(\frac{1}{r^{3/4}} \text{ means } \frac{1}{\sqrt[3]{r^4}}\right)$ — exist and continuous $\forall r > 0$, $\alpha < \theta < \alpha + 2\pi$

Cauchy - Riemann equs: $\forall r > 0$, $\alpha < \theta < \alpha + 2\pi$

$$r u_r = v_\theta = \frac{2}{4}\sqrt[4]{r}\cos\frac{\theta}{4}$$

$$u_\theta = -r v_r = -\frac{2}{4}\frac{r}{r^{3/4}}\sin\frac{\theta}{4} = -\frac{2}{4}\sqrt[4]{r}\sin\frac{\theta}{4}, \quad \forall r > 0, \ \alpha < \theta < \alpha + 2\pi$$

$$\Rightarrow f'(z) = e^{-i\theta}(u_r + i v_r)$$

$$= e^{-i\theta}\left(\frac{2}{4}\frac{1}{r^{3/4}}\cos\frac{\theta}{4} + i\frac{2}{4}\frac{1}{r^{3/4}}\sin\frac{\theta}{4}\right)$$

$$= e^{-i\theta + i\frac{3}{4}\theta}\frac{2}{4r^{3/4}} = \boxed{\frac{2}{4z^{3/4}}}$$ — exists $\forall z \neq 0$