Exam 2 solutions math 313

1. Find if $u(x,y) = 4xy - 2x$ is harmonic in some domain and if it is harmonic find a harmonic conjugate $v(x,y)$.

$u_{xx} + u_{yy} = 0 + 0 = 0 \Rightarrow$ harmonic in $\mathbb{C}$.

$v_x = -u_y = -4x$

$\Rightarrow v = -\int u_y \, dx = -2x^2 + \phi(y)$

$v_y = u_x$

$\Rightarrow \phi'(y) = 4y - 2$

$\Rightarrow \phi = 2y^2 + C - 2y$

$v = -2x^2 + 2y^2 + C - 2y$

2. Find the principle value of $(1+i\sqrt{3})^i$.

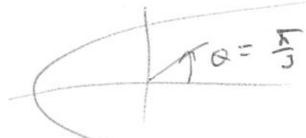
$$(1+i\sqrt{3})^i = e^{i\log(1+i\sqrt{3})}$$

principle value:

$$(1+i\sqrt{3})^i = e^{i\,\mathrm{Log}(1+i\sqrt{3})}$$

$$-\pi < \arg(1+i\sqrt{3}) < \pi$$

$$\mathrm{Log}(1+i\sqrt{3}) = \ln\sqrt{1+3} + i\,\mathrm{Arg}(1+i\sqrt{3})$$



$$e^{i\frac{\pi}{3}} = \frac{1}{2}(1+i\sqrt{3})$$

$$= \ln 2 + i\frac{\pi}{3}$$

$$\Rightarrow \quad (1+i\sqrt{3})^i = e^{i\ln 2 - \frac{\pi}{3}}$$

3. Show that
$$\sin(z_1 + z_2) = \cos z_1 \sin z_2 + \sin z_1 \cos z_2.$$

$$\cos z_1 \sin z_2 + \sin z_1 \cos z_2$$

$$= \left(\frac{e^{iz_1} + e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2} - e^{-iz_2}}{2i}\right) + \left(\frac{e^{iz_1} - e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2} + e^{-iz_2}}{2}\right)$$

$$= \frac{e^{i(z_1+z_2)} + e^{-i(z_1-z_2)} - e^{i(z_1-z_2)} - e^{i(-z_1-z_2)}}{4i}$$

$$+ \frac{e^{i(z_1+z_2)} - e^{i(-z_1+z_2)} + e^{i(z_1-z_2)} - e^{i(-z_1-z_2)}}{4i}$$

$$= \frac{2e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{4i} = \sin(z_1 + z_2)$$

4. Find all values of $\sinh^{-1}(4i)$.

$$W = \sinh^{-1}(z)$$

$$\Rightarrow z = \sinh W = \frac{e^{W} - e^{-W}}{2}$$

$$e^{2W} - 2ze^{W} - 1 = 0$$

$$e^{W} = z + (z^2+1)^{1/2}$$

$$W = \log\left(z + (z^2+1)^{1/2}\right)$$

$$\Rightarrow \sinh^{-1}(4i) = \log\left(4i + (-16+1)^{1/2}\right)$$

$$= \log\left(4i + (-15)^{1/2}\right)$$

(A) $\log(4i + i\sqrt{15}) = \ln(4+\sqrt{15}) + \frac{i\pi}{2} + 2\pi n i$, $n \in \mathbb{N}$

(B) $\log(4i - i\sqrt{15}) = \ln(4-\sqrt{15}) + \frac{i\pi}{2} + 2\pi n i$, $n \in \mathbb{N}$.

But $\ln(4-\sqrt{15}) = \ln \frac{16-15}{4+\sqrt{15}} = -\ln(4+\sqrt{15})$

$$\Rightarrow \sinh^{-1}(4i) = \pm \ln(4+\sqrt{15}) + i\left(\frac{\pi}{2} + 2\pi n\right),$$

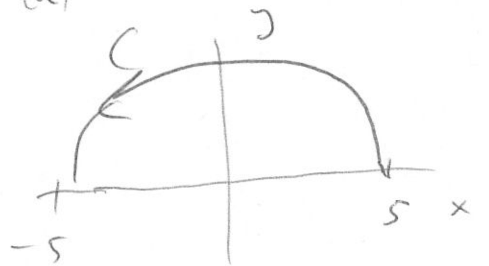
$n = 0, \pm 1, \pm 2, \ldots$

5. Assume that C is the positively oriented semicircle $z = 5e^{i\theta}$ $(0 \le \theta \le \pi)$ and $F(z)$ is the principle branch of $f(z) = z^{3-i}$.
(a) Use parametric representation for C to find

$$\int_C F(z)dz.$$

(b) What is a branch point for $f(z)$?

(a)



$f(z) = z^{3-i} = e^{(3-i)\log z}$

$\Rightarrow F(z) = e^{(3-i)\operatorname{Log} z}$

For $z = 5e^{i\theta}$

$\operatorname{Log} z = \ln 5 + i\theta, \quad -\pi < \theta < \pi$

$\Rightarrow \int_C F(z)dz = \int_C e^{(3-i)(\ln 5 + i\theta)} d(5e^{i\theta})$

$= \int_0^\pi 5i\, e^{(3-i)\ln 5} e^{3i\theta + \theta + i\theta} d\theta$

$= 5i\, e^{(3-i)\ln 5} \dfrac{e^{(4i+1)\theta}}{4i+1} \Big|_{\theta=0}^{\theta=\pi} = \dfrac{5i\, e^{(3-i)\ln 5}}{4i+1}[e^{\pi} - 1]$

$= \dfrac{(-4i+1)5i}{17} e^{(3-i)\ln 5}[e^{\pi} - 1] = \dfrac{5}{17}(i+4)e^{(3-i)\ln 5}[e^{\pi} - 1]$

(b) ~~Ant~~ branch point is a common point of all branch cuts $\Rightarrow$ $z = 0$ is a branch point.

6. Use antiderivative to find

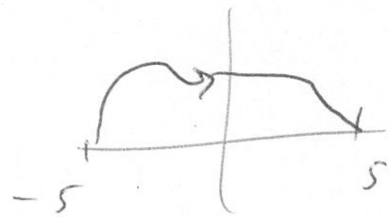
$$\int_{-5}^{5} F(z)dz,$$

where $F(z)$ is the principle branch of $f(z) = z^{3-i}$ and the integration contour connecting $z=-5$ and $z=5$ lies above the real axis except the end points.

$f(z) = e^{(3-i)\log z}$

Principle branch

$F(z) = e^{(3-i)\,\mathrm{Log}\, z}$



Antiderivative

$\hat{F}(z) = \dfrac{z^{4-i}}{4-i} = \dfrac{e^{(4-i)\,\mathrm{Log}\, z}}{4i}$

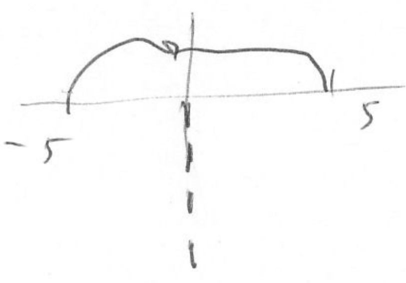
$\Rightarrow \displaystyle\int_{-4}^{4} F(z)\,dz = \hat{F}(z)\Big|_{z=-5}^{z=5}$

$= \dfrac{e^{(4-i)\,\mathrm{Log}\, z}}{4-i}\Bigg|_{z=-5}^{z=5}$

$= \dfrac{e^{(4-i)\ln 5}}{4-i} - \dfrac{e^{(4-i)(\ln 5 + i\pi)}}{4+i}$

$= \dfrac{e^{(4-i)\ln 5}}{4-i}\left[1 - e^{\pi}\right] = \dfrac{(4+i)\, e^{4\ln 5 - i\ln 5}\left[1-e^{\pi}\right]}{17} = \dfrac{(4+i)\, 5^4 e^{-i\ln 5}\left[1-e^{\pi}\right]}{17}$

For $z=-5$ we replace $\mathrm{Log}\, z$ with $\log z = \ln z + i\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

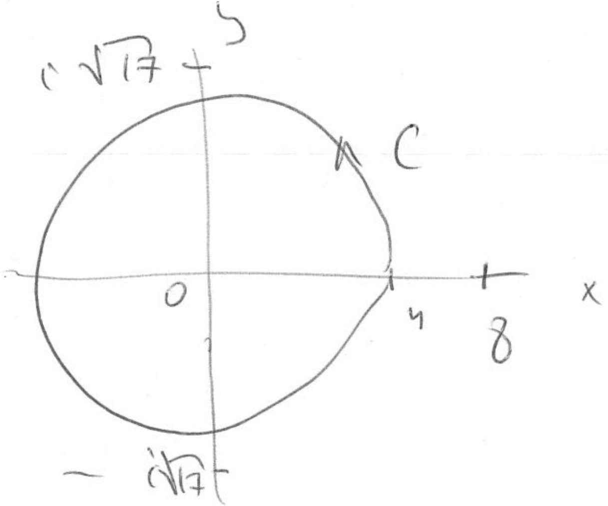


7. Find the value of

$$\int_C \frac{1}{(z-8)^2(z^2+17)} dz,$$

where C is the positively oriented circle $|z| = 4$.

Singularities of the rational function $f(z) = \dfrac{1}{(z-8)^2(z^2+17)}$ are determined by zeros of denominator: $z = 8$, $z = \pm i\sqrt{17}$



$\Rightarrow$ in interior to C $f(z)$ is analytic

$\Rightarrow$ by Cauchy–Goursat thm $\int_C f(z)\,dz = 0$.