math 313

HW 01 Solutions

p. 5

1 (a) $(\sqrt{2}-i)-i(1-\sqrt{2}i)$

$= \sqrt{2}-i-i+\sqrt{2}i^2 = -2i$

(b) $(2,-3)(-2,1) = (-4+3, 6+2)$

$= (-1, 8)$

(c) $(3,1)(3,-1)\left(\frac{1}{5}, \frac{1}{10}\right)$

$= (9+1, 3-3)\left(\frac{1}{5}, \frac{1}{10}\right) = (10,0)\left(\frac{1}{5}, \frac{1}{10}\right)$

$= (2, 1)$

2 (a) $Re(iz) = Re(ix-y) = -y = -Im(z)$

(b) $Im(iz) = I(ix-y) = x = Re\, z$

3 $(1+z)^2 = (1+z) + (1+z)z = 1+2z+z^2$

4 $z^2 = (1 \pm i)^2 = 1 \pm 2i - 1 = \pm 2i$

$z^2 - 2z + 2 = \pm 2i - 2(1 \pm i) + 2 = \pm 2i - 2 \mp 2i$

$+ 2 = 0$

⑤ ②

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$ (1)

$z_2 z_1 = (x_2 + iy_2)(x_1 + iy_1)$

$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$ (2)

(1) = (2) ∎

⑥ (a) $(z_1 + z_2) + z_3 = x_1 + x_2 + i(y_1 + y_2)$

$+ x_3 + iy_3 = x_1 + x_2 + x_3 + i(y_1 + y_2 + y_3)$

$= x_1 + (x_2 + x_3) + iy_1 + i(y_2 + y_3)$

$= z_1 + (z_2 + z_3)$ ∎

(b) $z_1(z_2 + z_3) = (x_1 + iy_1)(x_2 + iy_2 + x_3 + iy_3)$

$= x_1(x_2 + x_3) - y_1(y_2 + y_3) + i(y_1(x_2 + x_3)$

$+ x_1(y_2 + y_3)) = ((x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2))$

$+ ((x_1 x_3 - y_1 y_3 + i(y_1 x_3 + x_1 y_3)) = z_1 z_2 + z_1 z_3$

∎

p. 8

1

(a) $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$

$= \dfrac{(1+2i)(3+4i)}{9+16} + \dfrac{(2-i)(-5i)}{25}$

$= \dfrac{3-8+i(6+4)}{25} + \dfrac{-10i-5}{25} = \dfrac{-5+10i-10i-5}{25}$

$= -\dfrac{10}{25} = -\dfrac{2}{5}$

(b) $\dfrac{5i}{(1-i)(2-i)(3-i)} = \dfrac{5i}{(2-1+i(-2-1))(3-i)}$

$= \dfrac{5i}{(1-3i)(3-i)} = \dfrac{5i}{\cancel{3}-10i} = \dfrac{5i(10i)}{-10i \cdot 10i} = -\dfrac{1}{2}$

(c) $(1-i)^4 = ((1-i)^2)^2 = (1-2i-1)^2$

$= (-2i)^2 = -4$

② $z \neq 0$

$$\frac{1}{1/z} = \frac{1}{\bar{z}/|z|^2} = \frac{|z|^2}{\bar{z}} = \frac{|z|^2 z}{|z|^2} = z.$$

③ $(z_1 z_2)(z_3 z_4) = z_1(z_2 z_3) z_4$

$= z_1 (z_3 z_2) z_4 = (z_1 z_3)(z_2 z_4)$

④ $z_1 z_2 z_3 = (z_1 z_2) z_3 = 0$

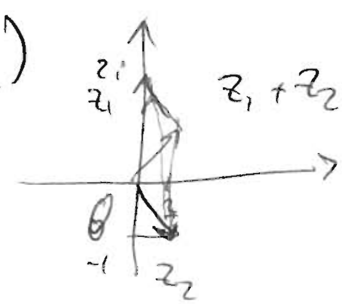
$\Rightarrow$ $z_1 z_2 = 0$ or $z_3 = 0$

$\Downarrow$

$z_1 = 0$ or $z_2 = 0$

$\Rightarrow$ $z_1 = 0$ or $z_2 = 0$ or $z_3 = 0$

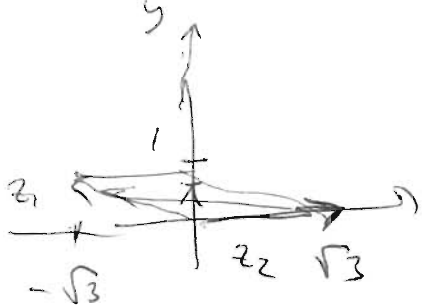
p12 ① a)



$z_1 + z_2 = \frac{2}{3} + i$

$z_1 - z_2 = -\frac{2}{3} + 3i$

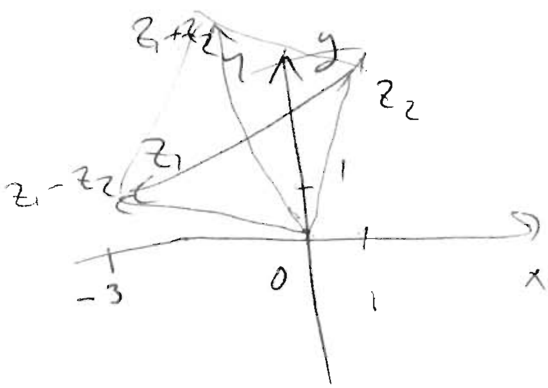
(b)



$z_1+z_2=(0,1)$

$z_1-z_2=(-2\sqrt{3},1)$

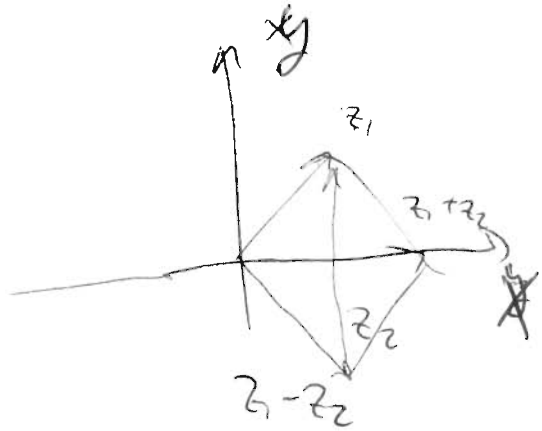
(c)



$z_1+z_2=(-2,5)$

$z_1-z_2=(-4,-3)$

(d)



$z_1+z_2=2x_1$

$z_1-z_2=2iy_1$

2

$\operatorname{Re} z = x \le |x| = |\operatorname{Re} z| \le \sqrt{x^2+y^2} = |z|$

$\operatorname{Im} z = y \le |y| = |\operatorname{Im} z| \le \sqrt{x^2+y^2} = |z|$

3

$|z_3+z_4| \ge ||z_3|-|z_4||$ $\qquad z_3+z_4$

$\Rightarrow \dfrac{1}{|z_3+z_4|} \le \dfrac{1}{||z_3|-|z_4||}$

$$\operatorname{Re}(z_1+z_2) \leq |z_1+z_2| \leq |z_1|+|z_2|$$

$$\Rightarrow \frac{\operatorname{Re}(z_1+z_2)}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}$$

(4) $$(|x|-|y|)^2 \geq 0$$

But $$(|x|-|y|)^2 = x^2+y^2-2|x||y| \geq 0$$

$$\Rightarrow 2|z|^2-|z|^2-2|x||y| \geq 0$$

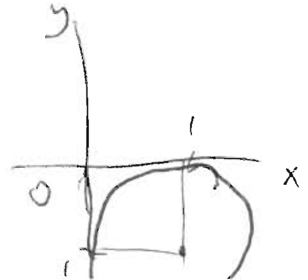
$$\Rightarrow 2|z|^2 \geq |z|^2+2|x||y|$$

$$= x^2+y^2+2|x||y| = (|x|+|y|)^2$$

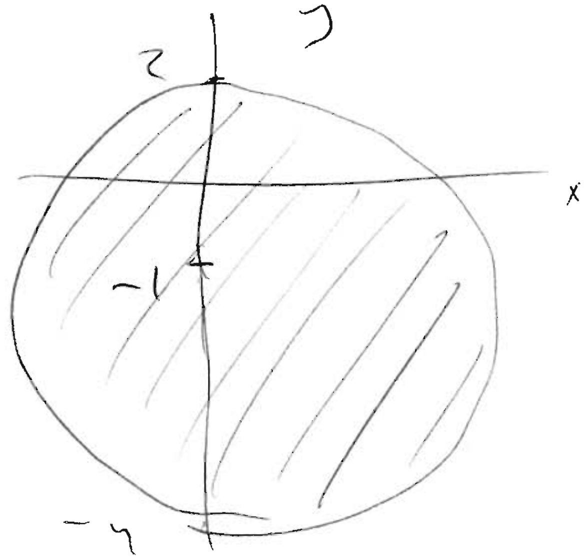
$$\Rightarrow \sqrt{2}\,|z| \geq ||x|+|y|| = |x|+|y|$$ $\boxtimes$

(5) (a) $|z-1+i|=1$

$|z-(1-i)|=1$ – circle of radius 1 centered at $(1-i$



(b) $|z+i| \le 3$ - disc of radius 3 centered at $-i$

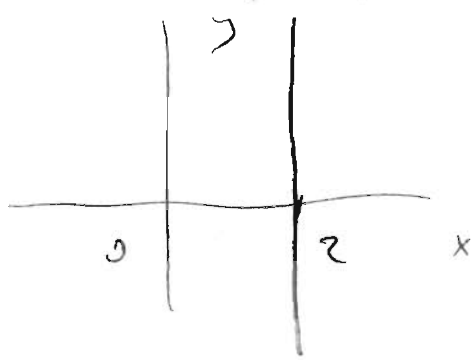


p.14 1 (a) $\overline{\bar{z}+3i} = \bar{\bar{z}} + \overline{3i}$

$= z - 3i$

(c) $\overline{(z+i)^2} = (2-i)^2 = 4-1-4i = 3-4i$

2 (a) $Re(\bar{z}-i) = Re(x-iy-i) = x = 2$



$$|Re(z+\bar{z}+z^3)|$$

$$\leq |z+\bar{z}+z^3| \leq 2+|z|+|z|^3$$

(since $|z|<1$
$\Rightarrow |z|^3<1$

$$\leq 2+1+1=4 \quad \boxed{}$$

11 (a)

1 $n=2 \Rightarrow \overline{z_1+z_2} = x_1+x_2-i(y_1+y_2)$

$$= \bar{z}_1+\bar{z}_2$$

2 assume for $n=m$:

$$\overline{z_1+\ldots+z_m} = \bar{z}_1+\ldots+\bar{z}_m$$

$$\overline{z_1+z_2+\ldots+z_{m+1}} = \overline{z_1+z_2+\ldots+z_m}+\bar{z}_{m+1}$$

$$= \bar{z}_1+\ldots+\bar{z}_{m+1} \quad \boxed{}$$

(15)

show $|z_1+z_2| \leq |z_1|+|z_2|$

$$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1}+\overline{z_2})$$

$$= |z_1|^2+|z_2|^2+z_1\overline{z_2}+\overline{z_1}z_2$$

$$= |z_1|^2+|z_2|^2+2\,\mathrm{Re}(z_1\overline{z_2})$$

$$\leq |z_1|^2+|z_2|^2+2|z_1||z_2| = (|z_1|+|z_2|)^2$$

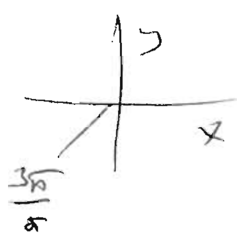
$$\Rightarrow |z_1+z_2| \leq |z_1|+|z_2| \quad \blacksquare$$

P.22 (1)

(a) $z = \frac{i}{-2-2i} = -\frac{1}{2}\,\frac{i}{1+i}$

$= -\frac{1}{2}\,\frac{i\,(1-i)}{2} = \frac{1}{4}(-1-i)$

$= \frac{1}{4}\, e^{i\left(-\frac{3}{4}\pi\right)} \Rightarrow \operatorname{Arg} z = -\frac{3\pi}{4}$



(b) $z = (\sqrt{3} - i)^6$

$\sqrt{3} - i = \sqrt{4}\, e^{i\theta}, \quad \theta = \arctan\left(-\frac{1}{\sqrt{3}}\right)$

$= -\frac{\pi}{6}$

$\Rightarrow \operatorname{Arg}(\sqrt{3} - i) = -\frac{\pi}{6}$

$z = \sqrt{4}^{6} \cdot e^{-i\frac{\pi}{6}\cdot 6} \Rightarrow \operatorname{Arg} z = -\pi + 2\pi = \pi.$

(3) Show $e^{i\theta_1} \cdots e^{i\theta_n} = e^{i(\theta_1 + \cdots + \theta_n)}$

(1) $n = 2$ $\quad e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$

from section (8).

(2) Assume for $m = n$ $\quad e^{i\theta_1} \cdots e^{i\theta_m} = e^{i(\theta_1 + \cdots \theta_m)}$

$$e^{i\theta_1} \cdots e^{i\theta_m} e^{i\theta_{m+1}}$$

$$= e^{i(\theta_1 + \cdots + \theta_m)} e^{i\theta_{m+1}} = e^{i(\theta_1 + \cdots \theta_{m+1})}$$

$\boxed{Z}$

(5) (a) $(1 - i\sqrt{3}) = 2e^{-i\frac{\pi}{3}}$, $\sqrt{3} + i = 2e^{i\frac{\pi}{6}}$,

$i = e^{i\frac{\pi}{2}}$

$$i(1 - i\sqrt{3})(\sqrt{3} + i) = 4e^{i\left(\frac{\pi}{2} - \frac{\pi}{3} + \frac{\pi}{6}\right)} = 4e^{i\frac{\pi}{3}}$$

$$= 4\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + 2(1 + i\sqrt{3})$$ 12

(c)

$$-1 + i = \sqrt{2}\, e^{i\frac{3}{4}\pi}$$

$$(-1+i)^7 = (\sqrt{2})^7 \cdot e^{i\frac{21}{4}\pi}$$

$$= (\sqrt{2})^7 \cdot e^{-i\frac{3}{4}\pi} = (\sqrt{2})^7 \frac{-1-i}{\sqrt{2}} = \underline{-8(1+i)}.$$

8 (optional) Assume that $|z_1| = |z_2|$

$$z_1 = r_1 e^{i\theta_1}$$
$$z_2 = r_2 e^{i\theta_2}$$
$$|z_1| = |z_2| \Rightarrow r_1 = r_2$$

choose $c_1 = \sqrt{r_1}\, e^{i\frac{\theta_1+\theta_2}{2}}$

$$c_2 = \sqrt{r_1}\, e^{i\frac{\theta_1-\theta_2}{2}}$$

$$\Rightarrow c_1 c_2 = r_1 e^{i\theta_1} = z_1$$

$$c_1 \bar{c}_2 = r_1 e^{i\theta_2} = z_2,$$ i.e. such c_1, c_2

Assume

$z_1 = c_1 c_2$

$z_2 = c_1 \overline{c_2}$

$\Rightarrow |z_1| = |c_1||c_2| = |z_2|$ ∎.