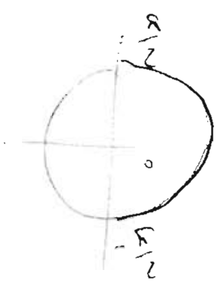
math 313

HW 02 Solutions

BC p 24 (6) $\operatorname{Re} z_1 > 0$, $\operatorname{Re} z_2 > 0 \Rightarrow$



$$\left.\begin{array}{l} \Rightarrow -\frac{\pi}{2} < \operatorname{Arg} z_1 < \frac{\pi}{2} \\ -\frac{\pi}{2} < \operatorname{Arg} z_2 < \frac{\pi}{2} \end{array}\right\} \Rightarrow \begin{array}{l} \operatorname{Arg}(z_1 z_2) \\ = \operatorname{Arg}(z_1) + \operatorname{Arg} z_2 + 2\pi n, \end{array}$$

where n such that

$$-\pi < \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) + 2\pi n < \pi$$

But $-\pi < \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) < \pi \Rightarrow n = 0$.

(11) (optional)

(a) Binominal formula:

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}, \quad n = 1, 2, \ldots$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad k = 0, 1, \ldots$$

(b) Moivre formula:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad n = 0, \pm 1, \ldots$$

From (a) and (b) for $z_2 = \cos\theta$, $z_1 = i\sin\theta \Rightarrow$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$= \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta\,(i\sin\theta)^k \quad , n = 0, 1, 2, \ldots$$

~~Rf~~

$$\Rightarrow \cos n\theta = Re(\cos n\theta + i\sin n\theta)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta\,(\sin\theta)^k\, Re(i^k)$$

If k is odd $\Rightarrow Re(i^k) = 0$

~~Rf~~ omitting all odd k

$$\Rightarrow \cos n\theta = \sum_{k=0}^{m} \binom{n}{2k} \cos^{n-2k}\theta\,(\sin\theta)^{2k}(i)^{2k},$$

where $m = \begin{cases} n/2 & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$

But $(i)^{2k} = (-1)^k$

$$\Rightarrow \cos n\theta = \sum_{k=0}^{m} \binom{n}{2k} \cos^{n-2k}\theta\,(\sin\theta)^{2k}(-1)^k.$$

(b) $x = \cos\theta$

$\sin^2\theta = 1 - \cos^2\theta = 1 - x^2$

$$\Rightarrow (\sin\theta)^{2k} = (\sin^2\theta)^k = (1 - x^2)^k$$

$$\Rightarrow \cos n\theta = \sum_{k=0}^{m} \binom{n}{2k} (-1)^k x^{n-2k} (1 - x^2)^k$$

p 29 ①

(a) $(2i)^{1/2}$

$$2i = 2e^{i\frac{\pi}{2} + 2k\pi i}, \quad k = 0, 1$$

$$\Rightarrow (2i)^{1/2} = \sqrt{2}\, e^{i\frac{\pi}{4} + \pi i k} = \pm \sqrt{2}\, e^{i\frac{\pi}{4}}$$

$$= \pm \sqrt{2}\, \frac{1+i}{\sqrt{2}} = \pm(1+i)$$

(b) $$1 - \sqrt{3}\, i = \sqrt{1+3}\, e^{-i\frac{\pi}{3} + 2\pi i k}, \quad k = 0, 1$$

$$\Rightarrow (1 - \sqrt{3}\, i)^{1/2} = \sqrt{2}\, e^{-i\frac{\pi}{6} + \pi i k} = \pm\sqrt{2}\, e^{-i\frac{\pi}{6}}$$

$$= \pm\sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \pm\left(\frac{\sqrt{3}}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

2 (a) $(-16)^{1/4} = \left(16 \cdot e^{i\pi + 2\pi i k}\right)^{1/4}$

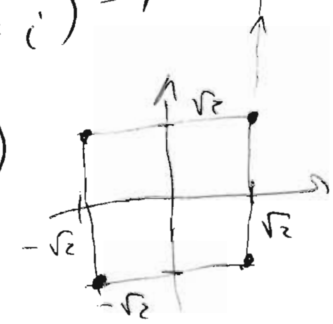
$= 2 \cdot e^{i\frac{\pi}{4} + \frac{\pi i k}{2}}$, $k = 0, 1, 2, 3$

$k=0$: $2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(1+i)$ – principle root

$k=1$: $2\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(-1+i)$

$k=2$: $2\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \sqrt{2}(-1-i)$

$k=3$: $2\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(1-i)$

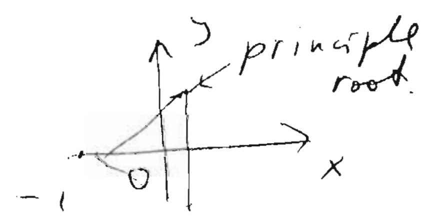


3 (a) $(-1)^{1/3} = \left(1 \cdot e^{i\pi}\right)^{1/3} = 1 \cdot e^{i\frac{\pi}{3} + \frac{2\pi}{3}k}$, $(k = 0, 1, 2)$

$k=0$: $\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$k=1$: $e^{i\left(\frac{\pi}{3} + \frac{2\pi}{3}\right)} = -1$

$k=2$: $e^{i\left(\frac{\pi}{3} + \frac{4\pi}{3}\right)} = e^{i\frac{5}{3}\pi} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$



4 $\quad \omega_3 = e^{i\frac{2\pi}{3}} = \frac{-1+\sqrt{3}i}{2}$

$z_0 = -4\sqrt{2} + 4\sqrt{2}\,i = 8\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = 8e^{i\frac{3}{4}\pi}$

$z_0^{1/3} = 2e^{i\frac{\pi}{4} + \frac{2\pi i k}{3}}$, $k = 0, 1, 2$

$\Rightarrow c_0 = 2e^{i\frac{\pi}{4}} = \sqrt{2}(1+i)$

$$C_1 = \omega_3 \cdot C_0 = e^{i\frac{2\pi}{3}} \cdot 2e^{i\frac{\pi}{4}} = 2(1+i)\left(\frac{-1+\sqrt{3}i}{2}\right)$$

$$= \frac{1}{\sqrt{2}}\left(-1-\sqrt{3}-i+\sqrt{3}i\right)$$

$$= \frac{-(\sqrt{3}+1)+(\sqrt{3}-1)i}{\sqrt{2}}$$

$$C_2 = \omega_3^2 C_0 = \omega_3 C_1 = \frac{-1+\sqrt{3}i}{2} \cdot \frac{-(\sqrt{3}+1)+(\sqrt{3}-1)i}{\sqrt{2}}$$

$$= \frac{\sqrt{3}+1-3+\sqrt{3}+i\left(-3-\sqrt{3}-\sqrt{3}+1\right)}{2\sqrt{2}}$$

$$= \frac{2\sqrt{3}-2+i\left(-2-2\sqrt{3}\right)}{2\sqrt{2}} = \frac{\sqrt{3}-1+i(-1-\sqrt{3})}{\sqrt{2}}$$

5 (a) $z = a+i$

$|z| = \sqrt{a^2+1} = A$

$\alpha = \mathrm{Arg}(A+i)$

$z^{1/2} = \sqrt{A}\, e^{i\left(\frac{\alpha}{2}+\pi k\right)}$, $(k=0,1)$

$= \pm\sqrt{A}\, e^{i\frac{\alpha}{2}}$

(b) $\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$ $\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$

$\Rightarrow \cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}$ $\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$

$\pm\sqrt{A}\, e^{i\frac{\alpha}{2}} = \pm\sqrt{A}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$

$a + i = A\cos\alpha + i A\sin\alpha$

$\Rightarrow \cos\alpha = \frac{a}{A}$ $\sin\alpha = \frac{1}{A}$

$\Rightarrow \pm\sqrt{A}\, e^{i\frac{\alpha}{2}} = \pm\sqrt{A}\left(\frac{\sqrt{1+\cos\alpha}}{\sqrt{2}} + i\frac{\sqrt{1-\cos\alpha}}{\sqrt{2}}\right)$

$= \frac{\pm\sqrt{A}}{\sqrt{2}}\left(\sqrt{1+\frac{a}{A}} + i\sqrt{1-\frac{a}{A}}\right) = \frac{\pm 1}{\sqrt{2}}\left(\sqrt{A+a} + i\sqrt{A-a}\right)$

7 (optional)

$1 + c + \ldots + c^{n-1} = \frac{1-c^n}{1-c} = 0$ because $c^n = 1$

8 (a) $az^2 + bz + c = 0$

$z^2 + \frac{2b}{2a}z + \frac{c}{a} = 0$

$z^2 + 2\frac{b}{2a}z + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$

$$\left(z+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\Rightarrow\quad z+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b+(b^2-4ac)^{1/2}}{2a}$$

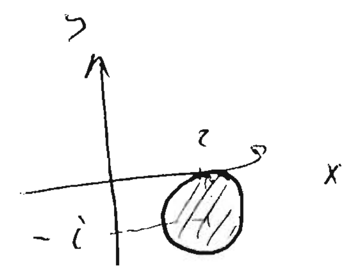
(b)

$$z^2+2z+(1-i)=0$$

$$z=\frac{-2+(4-4(1-i))^{1/2}}{2}=-1+(1-(1-i))^{1/2}$$

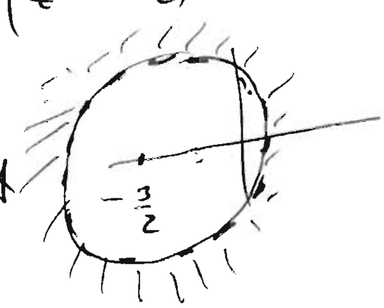
$$=-1+i^{1/2}=-1+e^{\frac{i\pi}{4}+\pi k}=-1\pm\frac{(1+i)}{\sqrt{2}}$$

p. 33 ① (a) $|z-2+i|\le 1$



closed set $\Rightarrow$ not a domain

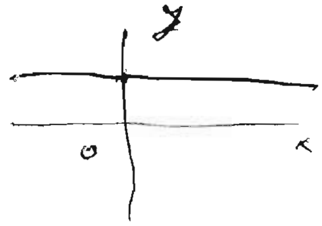
(b) $|2z+3|>4 \Rightarrow |z-(-\frac{3}{2})|>2$



open, connected, nonempty set

$\Rightarrow$ domain

(d) $\operatorname{Im} z = 1$



set is not open or closed if we include ∞ (contains boundary but not except ∞)

If neglect $\infty \Rightarrow$ closed set

$\Rightarrow$ not a domain in both cases.

(f) $|z-4| \geq |z|$

$|z-4|^2 \geq |z|^2$

$(x-4)^2 + y^2 \geq x^2 + y^2$

$\Rightarrow -8x + 16 \geq 0$

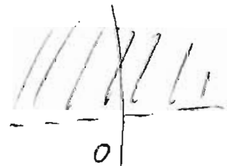
$x \leq 2$

$\Rightarrow$ closed set $\Rightarrow$ not a domain

② (a) closed

(b) open

(c) $\operatorname{Im} z > 1$



$\Rightarrow$ open set

(d) If include $\infty \Rightarrow$ neither open or closed. If disregard $\infty \Rightarrow$ ~~of~~ closed.

(e) $0 \leq \arg z \leq \frac{\pi}{4}, z \neq 0$

neither open or closed.

(f) closed set

3 (a) bounded

(b) unbounded (includ arbitrary large $|z|$)

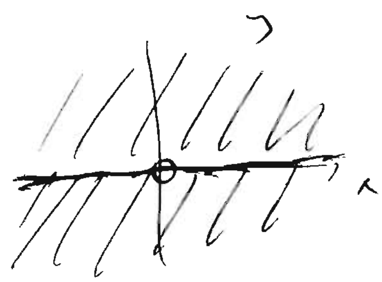
(c) unbounded

(d) unbounded

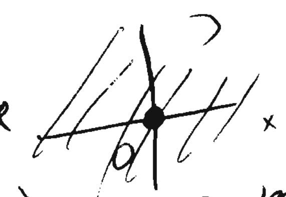
(e) unbounded

(f) unbounded.

4 (a) $-\pi < \arg z < \pi$, $z \neq 0$



$\Rightarrow$ closure



entire z-plane.

(c) $Re\left(\frac{1}{z}\right) \leq \frac{1}{2}$

$$Re\,\frac{1}{x+iy} = Re\,\frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} \leq \frac{1}{2}$$

$$x^2+y^2 \geq 2x$$

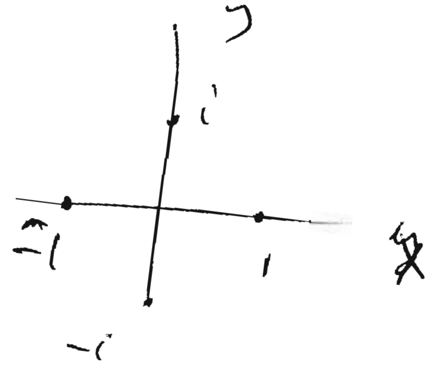
$$(x-1)^2+y^2-1 \geq 0 \Rightarrow |z-1| \geq 1$$

closure

7 (a) $z_n = i^n \quad (n = 1, 2, \ldots)$

$$\left.\begin{aligned} z_{4m} &= 1 \\ z_{4m+1} &= i \\ z_{4m+2} &= -1 \\ z_{4m+3} &= -i \end{aligned}\right\}, \quad m = 1, 2, \ldots$$

$\Rightarrow$ S

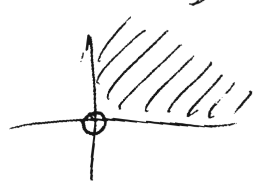


$\Rightarrow$ For any ~~neighb~~ deleted neighborhood of $i, -i, 1, -1$ we have no points of S $\Rightarrow$ not accumulation points.

For all other points z_0 we choose $\varepsilon > 0$ small enough so that $0 < |z - z_0| < \varepsilon$ does not contains $i, -1, 1, -1$ $\Rightarrow$ not accumulation points.

$\Rightarrow$ no accumulation points.

(c) $0 \le \arg z \le \frac{\pi}{2}, \ z \neq 0$



$\Rightarrow$ All points of set are accumulation points.

8) $\forall$ Boundary point is the accumulation point $\Rightarrow$ if all accumulation points are included $\Rightarrow$ all boundary points are included $\Rightarrow$ set is closed

9) (optional)

Domain is open nonemty connected set

$\Rightarrow$ take $z_0 \in S \Rightarrow$ z_0 is not a boundary point $\Rightarrow$ z_0 is interior point

$\Rightarrow \exists$ neigborhood containing only points of S

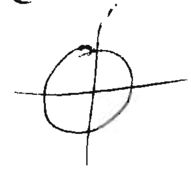
$\Rightarrow$ all smaller neighorhoods of that poin contains points from S $\Rightarrow$ accumulation point.

p. 37 1) (a) $f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$

$\Rightarrow$ domain of definition is all complex plane except $z = \pm i$

(d) $f(z) = \frac{1}{1 - z\bar{z}}$ $\Rightarrow |z|^2 \neq 1$

$\Rightarrow z \neq e^{i\theta}$ $-\pi < \theta \leq \pi$



$\Rightarrow$ all plane except ring $|z|=1$

2

$$z^3+z+1 = (x+iy)+x+iy+1$$

$$= x^3 - iy^3 + 3x^2iy - 3xy^2 + x + iy + 1$$

$$= (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$$

4 $$f = z + \frac{1}{z} = \cancel{x+iy+\frac{x-iy}{x^2+y^2}}$$

$$= re^{i\theta} + \frac{1}{r}e^{-i\theta} = \left(r+\frac{1}{r}\right)\cos\theta + i\left(r-\frac{1}{r}\right)\sin\theta$$