HW 03 Solutions

p 44

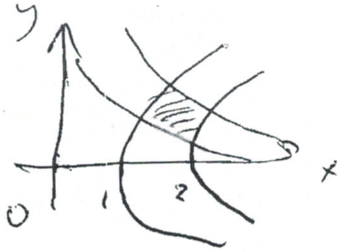
① $x^2 - y^2 + i2xy = 1 \Rightarrow x^2 - y^2 = 1$
$2xy = 0$

$$y = \pm\sqrt{x^2 - 1}, \quad x \geq 1$$

$x^2 - y^2 + 2ixy = 2 \Rightarrow y = \pm\sqrt{x^2 - 2}$

$x^2 - y^2 + 2ixy = i \Rightarrow y = \frac{1}{2x}$

$x^2 - y^2 + 2ixy = 2i \Rightarrow y = \frac{1}{x}$



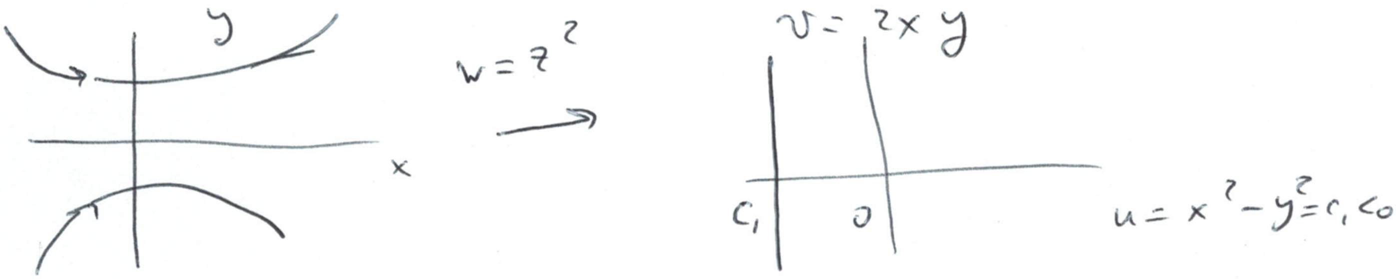
---

② $x^2 - y^2 = c_1 \quad (c_1 < 0)$

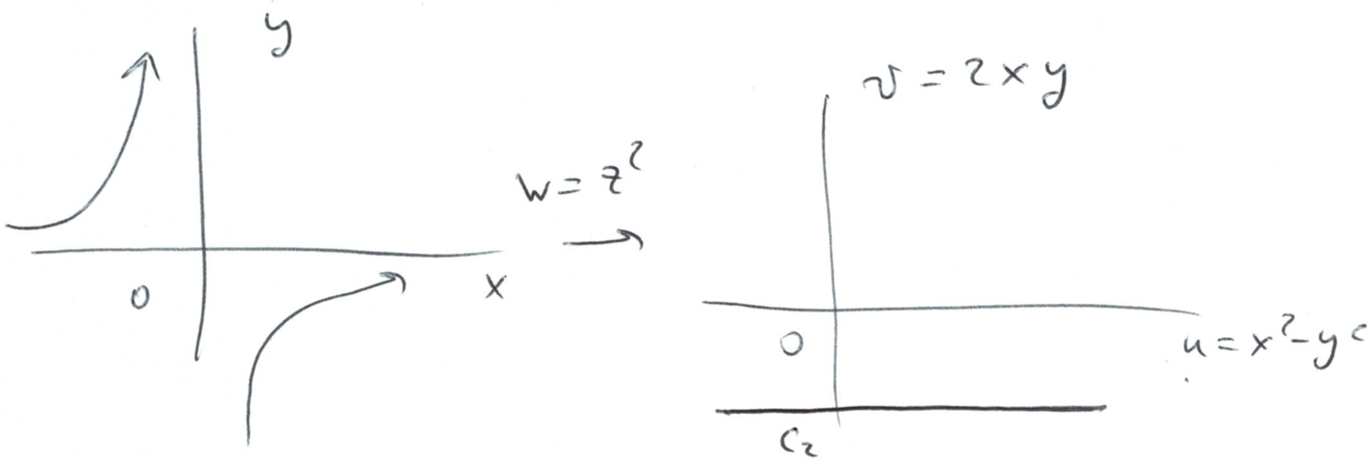
$w = z^2 = x^2 - y^2 + 2ixy$

$x^2 - y^2 = c_1 < 0$

$\Rightarrow y = \pm\sqrt{x^2 - c_1} = \pm\sqrt{x^2 + |c_1|}$



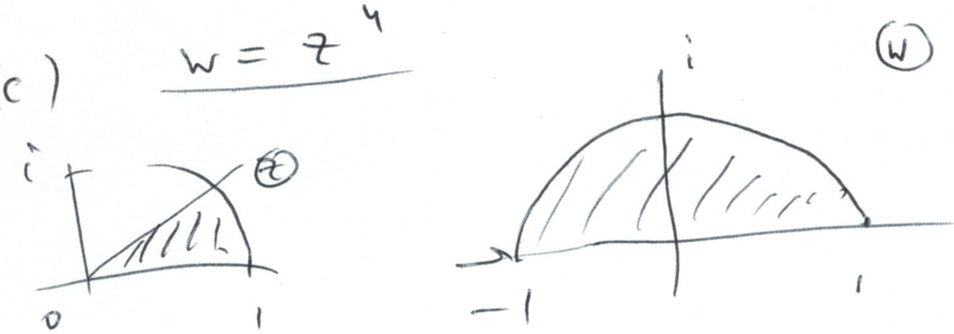
$2xy = c_2 < 0$



(3) (a) $w = z^2$ $r \leq 1,\ 0 \leq \theta \leq \frac{\pi}{4}$

(b) $w = z^3$

(c) $w = z^4$



P. 44 math 3 B (1)

(5) Horizontal segments

$z_1 = x + ic$

$z_2 = x + id$

$e^{z_1} = e^x e^{ic}$

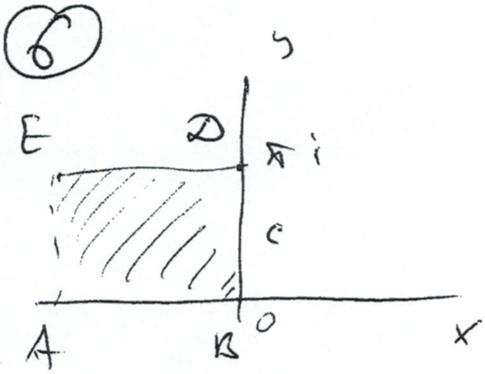
$e^{z_2} = e^x e^{id}$

$\Rightarrow e^a \le e^x \le e^b$

$\arg e^{z_1} = c \quad \arg e^{z_2} = d$

$\Rightarrow c \le \arg z \le d$

(6)



$w = e^z$

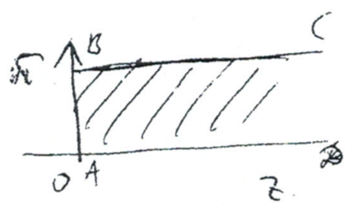
$B : z_B = 0 \Rightarrow w_{B'} = e^0 = 1$

$C' \Rightarrow$ segment$_{[0, i\pi]}$ BD : $z = e^{i\varphi} \Rightarrow$ circle C'

AB: $z = x \Rightarrow w = e^x \Rightarrow A'B'$ - real line between 1 at B' and $e^x \to 0$ at A' as $x \to -\infty$

ED $z = x + i\pi \Rightarrow w = e^{x+i\pi} = -e^x$ - real line between -1 and E' with $e^x \to 0$ as $x \to -\infty$

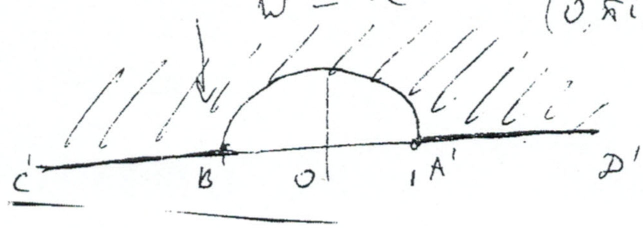
⑦



$x \geq 0, \quad 0 \leq y \leq \pi$

$w = e^z$

$(0, \pi i)$ transforms into $1 \leq \rho \leq \infty$



p.55

1 (a) Prove that $\lim_{z \to z_0} \mathrm{Re}(z) = \mathrm{Re}(z_0)$

Take $\delta = \varepsilon > 0$. Assume that $|z - z_0| < \delta$.

then $|\mathrm{Re}(z) - \mathrm{Re}(z_0)| \leq |z - z_0| < \varepsilon$ ∎

(b) Prove that $\lim_{z \to z_0} \bar{z} = \bar{z}_0$

Take $\delta = \varepsilon > 0$

Assume that $|z - z_0| < \delta$

then $|\bar{z} - \bar{z}_0| = |z - z_0| < \varepsilon$ ∎

2 (a) Show that

$$\lim_{z \to z_0} (az + b) = az_0 + b$$

Take $\delta = \frac{\varepsilon}{|a|}$ for $|a| \neq 0$. Assume that $|z - z_0| < \delta$.

then $|az + b - az_0 - b| = |a|\,|z - z_0| < |a|\delta = \varepsilon$.

For $a = 0$ limit is trivial. ∎

(b) Show that $\lim_{z \to z_0} (z^2 + c) = z_0^2 + c$

Assume that $|z - z_0| < \delta$

then

$$|z^2+c-z_0^2-c| = |z^2-z_0^2| = |(z-z_0)(z+z_0)|$$

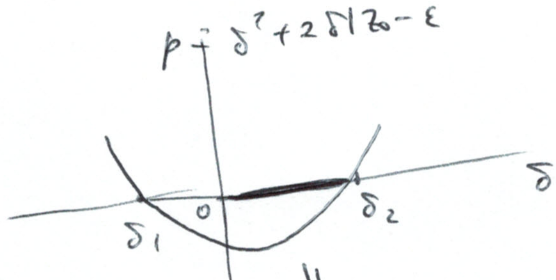
$$= |z-z_0|\,|z-z_0+2z_0| \leq |z-z_0|\left(|z-z_0|+2|z_0|\right)$$

$$< \delta(\delta+2|z_0|) < \varepsilon$$

we have to find δ such that this inequality is valid, i.e. $\delta^2+2\delta|z_0|-\varepsilon<0$

Solution ~~s~~ of quadratic equation for δ : $p = \delta^2+2\delta|z_0|-\varepsilon=0$

$$\delta_{1,2} = -|z_0| \pm \sqrt{|z_0|^2+\varepsilon}$$



$\Downarrow$

to ensure $p<0 \Rightarrow 0<\delta<\delta_2$

$$\delta_2 = -|z_0|+\sqrt{|z_0|^2+\varepsilon}$$

So we choose any $\delta>0$ s.t.

$$\delta < -|z_0|+\sqrt{|z_0|^2+\varepsilon}$$

$\blacksquare$

(5)

(3) (b) $\lim_{z\to i}\frac{iz^3-1}{z+i} = \frac{\lim_{z\to i}(iz^3-1)}{\lim_{z\to i}(z+i)} = \frac{-i^2-1}{2i} = \frac{0}{2i} = 0$.

(4) $\lim_{z\to z_0} z^n = z_0^n$

Pf by induction

1) For $n=1$ $\lim_{z\to z_0} z = z_0$

2) Assume $\lim_{z\to z_0} z^m = z_0^m$

$\Rightarrow \lim_{z\to z_0} z^{m+1} = \lim_{z\to z_0} z^m \lim_{z\to z_0} z = z_0^m \cdot z_0 = z_0^{m+1}$ ∎.

(5) Approaching along $z = x+i0$

$\Rightarrow \left(\frac{z}{\bar{z}}\right)^2 = \left(\frac{x}{x}\right)^2 = 1$

Approaching along $z = x+ix$

$\Rightarrow \left(\frac{z}{\bar{z}}\right)^2 = \left(\frac{x+ix}{x-ix}\right)^2 = \left(\frac{e^{i\frac{\pi}{4}}}{e^{-i\frac{\pi}{4}}}\right)^2 = -1$

$\Rightarrow$ limit does not exist.

9 Show that

$$\lim_{z \to z_0} f(z) g(z) = 0 \quad \text{if} \quad \lim_{z \to z_0} f(z) = 0$$

and $\exists M > 0$ s.t. $|g(z)| \leq M$ for $\forall z$ provided $|z - z_0| < \delta_1$ for some $\delta_1 > 0$.

<u>Pf</u>. $\lim_{z \to z_0} f(z) = 0$ implies that

$\forall \frac{\varepsilon}{|M|} > 0$ $\exists \delta > 0$ such that $|z - z_0| < \delta$ implies that $|f(z)| < \frac{\varepsilon}{|M|}$.

Assume that $|z - z_0| < \delta$ and $\delta < \delta_1$.

then $|f(z) g(z)| = |f(z)||g(z)| < \frac{\varepsilon}{M} |g(z)|$

$$\leq \frac{\varepsilon M}{M} = \varepsilon \quad \blacksquare$$

10 (a) $$\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = \left\{ \begin{array}{l} \text{by thm} \\ \text{of Sect 17} \end{array} \right\}$$

$$= \frac{\lim_{z\to 0} \frac{4}{z^2} \frac{1}{\left(\frac{1}{z}-1\right)^2}}{1} = \cancel{\lim_{z\to 0} \frac{\frac{4}{z^2}}{\frac{1}{z}-1}}$$

$$= \lim_{z\to 0} \frac{4}{(1-z)^2} = \frac{4}{1} = 4$$

(c) $$\lim_{z\to\infty} \frac{z^2+1}{z-1} = \lim_{z\to 0} \frac{\frac{1}{z^2}+1}{\frac{1}{z}-1} = \frac{\left(\lim_{z\to 0}\frac{1}{z}\right) \lim_{z\to 0}(1+z^2)}{\lim_{z\to 0} 1-z}$$

$$= \lim_{z\to 0} \frac{1}{z} = \infty$$