math 313 hw 04 sols

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1 (a) $f(z) = 3z^2 - 2z + 4$

$f'(z) = 3(z^2)' - 2(z') + (4)' = 6z - 2$

(c)

$f(z) = \dfrac{z-1}{2z+1}$, $z \neq \frac{1}{2}$

$f'(z) = \dfrac{1(2z+1) - (z-1)2}{(2z+1)^2} = \dfrac{3}{(2z+1)^2}$

(d) $f(z) = \frac{(1+z^2)^4}{z^2}, \quad z \neq 0$

$f'(z) = \frac{2z4(1+z^2)^3 z^2 - (1+z^2)^4 2z}{z^4} = \frac{8(1+z^2)^3}{z}$

$- 2\frac{(1+z^2)^4}{z^3}$

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a) $p(z) = a_0 + a_1 z + .. + a_n z^n, \; a_n \neq 0$

Show by induction that

$p'(z) = a_1 + 2a_2 z + .. + n a_n z^{n-1}$

(1) $n = 1$

$\Rightarrow P'(z) = (a_0 + a_1 z)' = a_1$

(2) Assume for $n = m$ $\quad p'(z) = a_1 + 2a_2 z + .. + m a_m z^{m-1}$

Show for $n = m+1$:

$p'(z) = (a_0 + a_1 z + .. + a_m z^m + a_{m+1} z^{m+1})'$

$= (a_0 + .. + a_m z^m)' + (a_{m+1} z^{m+1})'$

$= a_1 + 2a_2 z + .. + m a_m z^{m-1} + (m+1) a_{m+1} z^m$ ⊠

(b) Show by induction that
$$a_0 = p(0),\ a_1 = \frac{p'(0)}{1!},\ \dots\ a_n = \frac{p^{(n)}}{n!}$$

First show by induction that
$$(z^n)^{(k)} = n(n-1)\dots(n-k+1)z^{n-k}, \quad k \leq n$$

(1) $k=1$. $\quad (z^n)' = n z^{n-1}$

(2) Assume for $\underset{k\leq n}{k=m}$: $\quad (z^n)^{(k)} = n(n-1)\dots(n-k+1)z^{n-k}$

$$\Rightarrow \quad (z^n)^{(k+1)} = n(n-1)\dots(n-k+1)(z^{n-k})'$$

$$= n(n-1)\dots(n-(k+1)-1)\, z^{n-k-1}$$

$$\Rightarrow \quad (z^n)^{(k)}\Big|_{z=0} = 0 \ \text{ if } k < z.$$

$$(z^n)^{(n)} = n!$$

$$(z^n)^{(m)} = \left((z^n)^{(n)}\right)^{(m-n)} = 0 \ \text{ if } m > n.$$

$$\Rightarrow \quad p^{(n)}(0) = n!\, a_n \Rightarrow a_n = \frac{p^{(n)}(0)}{n!} \ \forall n \geq 0$$

3 $\quad w = \frac{1}{z}, \ z \neq 0$

$$w' = \lim_{\Delta z \to 0} \frac{\frac{1}{z+\Delta z} - \frac{1}{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{z - (z+\Delta z)}{z(z+\Delta z)\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-\Delta z}{z(z+\Delta z)\Delta z} = \lim_{\Delta z \to 0} \frac{-1}{z(z+\Delta z)} = -\frac{1}{z^2}$$

(6) (a) $(z^n)' = nz^{n-1}$.

show by induction

1) for $n=1$: $(z)' = 1$ – true.

2) Suppose that for $m=n$ $(z^m)' = mz^{m-1}$

then for $n = m+1$:

$$(z^{m+1})' = (z \cdot z^m)' = z' z^m + z(z^m)'$$

$$= z^m + m z \cdot z^{m-1} = (m+1) z^m$$

(8 (a)) $f(z) = \operatorname{Re} z$. Show that $f'(z)$ does not exist for any z.

If approaching z along Im axis:

$$\Delta z = i\Delta y$$

$$\lim_{\Delta y \to 0} \frac{\operatorname{Re}(z+i\Delta y) - \operatorname{Re} z}{i\Delta y} = \lim_{\Delta y \to 0} \frac{\operatorname{Re} z - \operatorname{Re} z}{i\Delta y} = 0$$

If approaching along Re axis:

by $\Delta z = \Delta x$

$$\lim_{\Delta x \to 0} \frac{\operatorname{Re}(z+\Delta x) - \operatorname{Re} z}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$\Rightarrow \lim\limits_{\Delta z \to 0} \dfrac{\Delta w}{\Delta z}$ does not exist and

f' is not differentiable $\forall z$.

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$$f(z) = \begin{cases} \dfrac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

$$\Delta w = f(z) - f(0) = \frac{\bar{z}^2}{z}$$

(a) $\Delta z = \Delta x$ – along real axis

$$\lim_{\Delta x \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta x^2}{(\Delta x)^2} = 1$$

(b) $\Delta z = i\Delta y$ – along imaginary axis

$$\lim_{\Delta y \to 0} \frac{\Delta w}{i\Delta y} = \lim_{\Delta y \to 0} \frac{(-i\Delta y)^2}{(i\Delta y)^2} = 1$$

(c) Along $\Delta x = \Delta y$
~~lim~~ $\Rightarrow \Delta z = \Delta x + i\Delta x$

$$\lim_{\Delta z \to 0} \frac{\bar{z}^2}{(\Delta z)(\Delta z)} = \lim_{\Delta x \to 0} \frac{(\Delta x - i\Delta x)^2}{(\Delta x + i\Delta x)^2}$$

$$= \lim_{\Delta x \to 0} \frac{(1-i)^2}{(1+i)^2} = \left(\frac{1-i}{1+i}\right)^2 = \frac{(1-i)^4}{2^2} = -1$$

$\Rightarrow f'(0)$ does not exist.

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(a) $f(z) = \bar{z} = x - iy$

$\Rightarrow u = x, \ v = -y$

$u_x = 1, \ v_y = -1 \Rightarrow \underline{u_x \neq v_y}$

$\Rightarrow f'(z)$ does not exist $\forall z$.

(b) $f(z) = z - \bar{z} = 2iy, \ u = 0, v = 2y$

$u_x = 0, \ v_y = 2 \Rightarrow u_x \neq v_y$

(c) $f(z) = 2x + ixy^2 \Rightarrow u = 2x, v = xy^2$

$u_x = 2, \ v_y = 2xy \Rightarrow u_x = v_y$ for $xy = 1$

$u_y = 0, \ v_x = y^2 \Rightarrow u_y = -v_x$ for $y = 0$

these conditions cannot be true simultaneously $\Rightarrow f'(z)$ does not exist $\forall z$.

(d) $f(z) = e^x e^{-iy} \Rightarrow u = e^x \cos y, \ v = -e^x \sin y$

$u_x = e^x \cos y, \ v_y = -e^x \cos y$

$\Rightarrow u_x = v_y$ i $e^x \cos y = -e^x \cos y$ only if $\cos y = 0$

or $y = \frac{\pi}{2} + \pi n, \ (1)$

$n = 0, \pm 1, \ldots$

$u_y = -e^x \sin y \qquad v_x = -e^x \sin y$

$\Rightarrow u_y = -v_x \Rightarrow -e^x \sin y = -(-e^x \sin y) \Rightarrow$

$\Rightarrow \sin y = 0 \Rightarrow y = \pi n, \; n = 0, \pm 1, \ldots$ (2)

Conditions (1) and (2) cannot be satisfied simultaneously $\Rightarrow f'(z)$ does not exist $\forall z$.

② (a) $\quad f(z) = iz + 2 = 2 + ix - y$

$u = 2 - y$

$v = x$

conditions of ~~thm 22~~ thm of sec 22

(1) $u_x = 0, \; u_y = -1$; $v_x = 1, \; v_y = 0$ — exist for $\forall x, y$ and continuous

(2) $u_x = v_y = 0$; ~~v_y~~ $u_y = -v_x = 1$ — Cauchy-Riemann equations are satisfied

$\Rightarrow$ by thm of Sect 22

$f'(z)$ exists $\forall z$

and $f'(z) = u_x + i v_x = i$

For $g = f'(z) = i$ ⑤

$u = 0,\ v = 1 \Rightarrow u_x = u_y = v_x = v_y = 0$

Conditions of thm of sec 22

(a) – exist and continuous $\forall z$.

(b) $u_x = v_y = 0$ – Cauchy – Riemann equations are
$u_y = -v_x = 0$ satisfied

$\Rightarrow f''(z)$ exists $\forall z$.

and $f''(z) = u_x + i v_x = 0$

~~2b $f(z) = x^2 + iy^2$~~

2d $f(z) = \cos x \cosh y - i \sin x \sinh y$

$u = \cos x \cosh y$

$v = -\sin x \sinh y$

(a) $u_x = -\sin x \cosh y$ exist and
$u_y = \cos x \sinh y$ continuous $\forall z$.
$v_x = -\cos x \sinh y$
$v_y = -\sin x \cosh y$

(b) $u_x = v_y = -\sin x \cosh y$ Cauchy-
$u_y = -v_x = \cos x \sinh y$ Riemann eqn. are satisfied

$\Rightarrow$ by th of Sec 22 $f'(z)$ exists $\forall z$

and $f'(z) = u_x + i v_x$

$= -\sin x \cosh y - i \cos x \sinh y$

Now for $g = f'(z) = -\sin x \cosh y - i \cos x \sinh y$

$u = -\sin x \cosh y$

$v = -\cos x \sinh y$

(1) $u_x = -\cos x \cosh y$
$u_y = -\sin x \sinh y$
$v_x = \sin x \sinh y$
$v_y = -\cos x \cosh y$
exist and continuous $\forall x, y$

(2) $u_x = v_y = -\cos x \cosh y$
$u_y = -v_x = -\sin x \sinh y$
Cauchy-Riemann equations are satisfied

$\Rightarrow$ by the [illegible] of sec 22. $g' = f''(z)$ exists $\forall z$.

and $f''(z) = g'(z) = u_x + i v_x$

$= -\cos x \cosh y + i \sin x \sinh y = -f(z)$

3 (a) $f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$

$$u = \frac{x}{x^2+y^2} \qquad , v = -\frac{y}{x^2+y^2}$$

1) $u_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$u_y = \frac{-2xy}{(x^2+y^2)^2}$

$v_x = \frac{2xy}{(x^2+y^2)^2}$

$v_y = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)} = \frac{y^2-x^2}{x^2+y^2}$

exist and continuous $\forall z \neq 0$

2) $u_x = v_y = \frac{y^2-x^2}{(x^2+y^2)^2}$

$u_y = -v_x = -\frac{2xy}{(x^2+y^2)^2}$

$\Rightarrow$ by thm of Sect 22 $f'(z)$ exists $\forall z \neq 0$

and

$$f'(z) = u_x + iv_x = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{2ixy}{(x^2+y^2)^2}$$

$$= -\frac{\bar{z}^2}{|z|^2} = -\frac{1}{z^2}$$

3 b) $f(z) = x^2 + iy^2$

$u = x^2$

$v = y^2$

(1) $\left.\begin{aligned} u_x &= 2x \\ u_y &= 0 \\ v_x &= 0 \\ v_y &= 2y \end{aligned}\right\}$ exist and continuous $\forall x, y$.

(2) $\left.\begin{aligned} u_x &= v_y \Rightarrow x = y \\ u_y &= -v_x = 0 \ \forall x, y \end{aligned}\right\} \Rightarrow x = y$ - line.

$\Rightarrow$ by thm of Sec 22 $f'(z)$ exists only along line $x = y$ ($\Rightarrow z = x + ix$)

and $f'(x+ix) = u_x + iv_x = 2x$.

4(b) $f(z) = \sqrt{r}\, e^{i\frac{\theta}{2}}$ $(r > 0,\ \alpha < \theta < \alpha + 2\pi)$

$u = \sqrt{r}\cos\frac{\theta}{2}$ $\quad v = \sqrt{r}\sin\frac{\theta}{2}$

(1) $u_r = \frac{1}{2}\frac{1}{\sqrt{r}}\cos\frac{\theta}{2}$, $v_r = \frac{1}{2}\frac{1}{\sqrt{r}}\sin\frac{\theta}{2}$ $\Big\}$

$u_\theta = -\frac{1}{2}\sqrt{r}\sin\frac{\theta}{2}$, $v_\theta = \frac{1}{2}\sqrt{r}\cos\frac{\theta}{2}$ } exist and continuous $\forall\, r>0, \theta$

2) $r u_r = v_\theta$, $u_\theta = -r v_r$ } these Cauchy - Riemann equations $\forall\, r>0$ and θ

$\Rightarrow$ by thm of Sec 23 $f'(z)$ exist $\forall\, r>0$, θ

and $f'(z) = e^{-i\theta}(u_r + i v_r)$

$$= e^{-i\theta}\left(\frac{1}{2}\frac{1}{\sqrt{r}}\cos\frac{\theta}{2} + i\frac{1}{2}\frac{1}{\sqrt{r}}\sin\frac{\theta}{2}\right)$$

$$= \frac{1}{2}\frac{1}{\sqrt{r}}e^{-i\theta}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = \frac{1}{2}\frac{1}{\sqrt{r}}e^{-i\frac{\theta}{2}}$$

$$= \frac{1}{2f(z)}$$

4c $f(z) = e^{-\theta}\cos(\ln r) + i e^{-\theta}\sin(\ln r)$

$r>0,\ 0<\theta<2\pi$

$u = e^{-\theta}\cos(\ln r)$, $v = e^{-\theta}\sin(\ln r)$

(1) $u_r = e^{-\theta}\frac{1}{r}(-\sin(\ln r))$ | $u_\theta = -e^{-\theta}\cos(\ln r)$ } exist and continuous $\forall\, r>0$.

$v_r = e^{-\theta}\frac{1}{r}\cos(\ln r)$ | $v_\theta = -e^{-\theta}\sin(\ln r)$

b)
$$\left.\begin{aligned} r u_r &= v_\theta \\ u_\theta &= -r v_r \end{aligned}\right\} \forall r>0$$

$\Rightarrow$ by thm of Sec 23 $f'(z)$ exist $\forall r>0$

and
$$f'(z) = e^{-i\theta}(u_r + i v_r)$$
$$= e^{-i\theta}\left(-\frac{1}{r}e^{-\theta}\sin(\ln r) + i\frac{1}{r}e^{-\theta}\cos(\ln r)\right)$$
$$= e^{-i\theta-\theta}\frac{1}{r}\, i\left(\cos(\ln r) + i\sin(\ln r)\right) = i\frac{e^{-i\theta-\theta}}{r}e^{i\ln r}$$
$$= \frac{i}{re^{i\theta}} f(z) = \frac{i f(z)}{z}.$$

6)
$$f(z) = \begin{cases} \dfrac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

$$f(z) = \frac{\bar{z}^2}{z} = \frac{\bar{z}^3}{x^2+y^2} = \frac{x^3 + iy^3 + 3x^2(-iy) - 3y^2 x}{x^2+y^2}$$
$$= \frac{x^3 - 3y^2 x}{x^2+y^2} + i\,\frac{y^3 - 3x^2 y}{x^2+y^2}$$

$$u = \begin{cases} \dfrac{x^3 - 3y^2 x}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases} \qquad v = \begin{cases} \dfrac{y^3 - 3x^2 y}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

derivatives

$$u(x,0) = \frac{x^3}{x^2} = x$$

$$\Rightarrow u_x(x,0) = 1$$

$$u(0,y) = 0$$

$$\Rightarrow u_y(0,y) = 0$$

$$v(x,0) = 0$$

$$\Rightarrow v_x(x,0) = 0$$

$$v(0,y) = \frac{y^3}{y^2} = y$$

$$\Rightarrow v_y(0,y) = 1$$

$$\Rightarrow \quad u_x\Big|_{\substack{x=0\\y=0}} = v_y\Big|_{\substack{x=0\\y=0}} = 1$$

$$u_y\Big|_{x=0,y=0} = -v_x\Big|_{x=0,y=0} = 0$$

— Cauchy – Riemann equations are satisfied but partial derivatives are not continuous $\Rightarrow$ $f'(0)$ does not exist.

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$$\begin{cases} u_r = u_x \cos\theta + u_y \sin\theta \\ u_\theta = -u_x r \sin\theta + u_y r \cos\theta \quad (*) \end{cases}$$

$\Rightarrow$ solve for ~~v_x, of u_x~~ ~~u_y~~ —

u_x, u_y ~~assum~~ as for unknowns:

$$r\cos\theta\, u_r = u_x \cos\theta \cdot r\cos\theta + u_y \sin\theta\cos\theta\, r$$

$$+ \quad -\sin\theta\, u_\theta = +u_x r \sin^2\theta \;\cancel{+}\; u_y r\cos\theta\sin\theta$$

$$\Rightarrow u_x \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} r = r\cos\theta\, u_r - u_\theta \sin\theta$$

$$\Rightarrow u_x = u_r \cos\theta - \frac{u_\theta}{r}\sin\theta \quad (**)$$

$\Rightarrow$ plug into $(*)$ to obtain:

$$u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r} \quad (***)$$

Similar,
$$\begin{cases} v_x = v_r \cos\theta - v_\theta \frac{\sin\theta}{r} \quad (\cancel{*}****) \\ v_y = v_r \sin\theta + v_\theta \frac{\cos\theta}{r} \quad (*****) \end{cases}$$

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Assume now that Cauchy-Riemann eqns. in polar form are satisfied:

$$r u_r = v_\theta \,, \quad u_\theta = -r v_r \quad (A)$$

$\Rightarrow$ plug in (A) into (* *)

$$\Rightarrow \quad u_x = v_\theta \frac{\cos\theta}{r} + v_r \sin\theta = v_y \text{ (from (** * * *))}$$

and $$u_y = v_\theta \frac{\sin\theta}{r} - v_r \cos\theta = -v_x \text{ (from (* * * *))}$$

8 (a) $f(z) = \text{Re}\, z \Rightarrow \frac{\Delta w}{\Delta z} = \frac{\Delta x}{\Delta z}$

For $\Delta z = \Delta x + i0$

$\Rightarrow \frac{\Delta w}{\Delta z} = 1$

For $\Delta z = 0 + i\Delta y$

$\Rightarrow \frac{\Delta w}{\Delta z} = \frac{0}{i\Delta y} = 0$

$\Rightarrow \lim \frac{\Delta w}{\Delta z} \nexists$

$\Rightarrow f'(z)$ does not exist.

9 (optional)

(a) Write $f(z)=u(r,\theta)+iv(r,\theta)$. Then recall the polar form

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

of the Cauchy-Riemann equations, which enables us to rewrite the expression (Sec. 23)

$$f'(z_0)=e^{-i\theta}(u_r+iv_r)$$

for the derivative of f at a point $z_0=(r_0,\theta_0)$ in the following way:

$$f'(z_0)=e^{-i\theta}\left(\frac{1}{r}v_\theta-\frac{i}{r}u_\theta\right)=\frac{-i}{re^{i\theta}}(u_\theta+iv_\theta)=\frac{-i}{z_0}(u_\theta+iv_\theta).$$

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(b) Consider now the function

$$f(z)=\frac{1}{z}=\frac{1}{re^{i\theta}}=\frac{1}{r}e^{-i\theta}=\frac{1}{r}(\cos\theta-i\sin\theta)=\frac{\cos\theta}{r}-i\frac{\sin\theta}{r}.$$

With

$$u(r,\theta)=\frac{\cos\theta}{r} \quad\text{and}\quad v(r,\theta)=-\frac{\sin\theta}{r},$$

the final expression for $f'(z_0)$ in part (a) tells us that

$$f'(z)=\frac{-i}{z}\left(-\frac{\sin\theta}{r}-i\frac{\cos\theta}{r}\right)=-\frac{1}{z}\left(\frac{\cos\theta-i\sin\theta}{r}\right)$$

$$=-\frac{1}{z}\left(\frac{e^{-i\theta}}{r}\right)=-\frac{1}{z}\left(\frac{1}{re^{i\theta}}\right)=-\frac{1}{z^2}$$

when $z\neq 0$.

10. (a) We consider a function $F(x,y)$, where

(optional)

$$x=\frac{z+\bar{z}}{2} \quad\text{and}\quad y=\frac{z-\bar{z}}{2i}.$$

Formal application of the chain rule for multivariable functions yields

$$\frac{\partial F}{\partial \bar{z}}=\frac{\partial F}{\partial x}\frac{\partial x}{\partial \bar{z}}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial \bar{z}}=\frac{\partial F}{\partial x}\left(\frac{1}{2}\right)+\frac{\partial F}{\partial y}\left(-\frac{1}{2i}\right)=\frac{1}{2}\left(\frac{\partial F}{\partial x}+i\frac{\partial F}{\partial y}\right).$$

(b) Now define the operator

$$\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right),$$

suggested by part (a), and formally apply it to a function $f(z)=u(x,y)+iv(x,y)$:

$$\frac{\partial f}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)=\frac{1}{2}\frac{\partial f}{\partial x}+\frac{i}{2}\frac{\partial f}{\partial y}$$

$$=\frac{1}{2}\left(u_x+iv_x\right)+\frac{i}{2}\left(u_y+iv_y\right)=\frac{1}{2}\left[\left(u_x-v_y\right)+i\left(v_x+u_y\right)\right].$$

If the Cauchy-Riemann equations $u_x=v_y, u_y=-v_x$ are satisfied, this tells us that $\partial f/\partial \bar{z}=0$.

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(a) $f(z) = \underbrace{3x+y}_{u} + i\underbrace{(3y-x)}_{v}$ is entire since

$$u_x = 3 = v_y \quad \text{and} \quad u_y = 1 = -v_x.$$

(b) $f(z) = \underbrace{\sin x \cosh y}_{u} + i\underbrace{\cos x \sinh y}_{v}$ is entire since

$$u_x = \cos x \cosh y = v_y \quad \text{and} \quad u_y = \sin x \sinh y = -v_x.$$

(c) $f(z) = e^{-y}\sin x - ie^{-y}\cos x = \underbrace{e^{-y}\sin x}_{u} + i\underbrace{(-e^{-y}\cos x)}_{v}$ is entire since

$$u_x = e^{-y}\cos x = v_y \quad \text{and} \quad u_y = -e^{-y}\sin x = -v_x.$$

(d) $f(z) = (z^2-2)e^{-x}e^{-iy}$ is entire since it is the product of the entire functions

$$g(z) = z^2 - 2 \quad \text{and} \quad h(z) = e^{-x}e^{-iy} = e^{-x}(\cos y - i\sin y) = \underbrace{e^{-x}\cos y}_{u} + i\underbrace{(-e^{-x}\sin y)}_{v}.$$

The function g is entire since it is a polynomial, and h is entire since

$$u_x = -e^{-x}\cos y = v_y \quad \text{and} \quad u_y = -e^{-x}\sin y = -v_x.$$

2. (a) $f(z) = \underbrace{xy}_{u} + i\underbrace{y}_{v}$ is nowhere analytic since

$$u_x = v_y \Rightarrow y = 1 \quad \text{and} \quad u_y = -v_x \Rightarrow x = 0,$$

which means that the Cauchy-Riemann equations hold only at the point $z = (0,1) = i$.

(b) $f(z) = 2xy + i(x^2 - y^2)$

$u = 2xy \qquad v = x^2 - y^2$

$u_x = 2y \qquad v_x = 2x$

$u_y = +2x \qquad v_y = -2y$

$\left.\begin{array}{l} u_x = v_y \Rightarrow 2y = -2y \Rightarrow y = 0 \\ u_y = -v_x \Rightarrow 2x = -2x \Rightarrow x = 0 \end{array}\right\} \Rightarrow$ Cauchy-Riemann eqns. hold only at $z = 0$.

$\Rightarrow$ not analytic.

(c) $f(z)=e^{y}e^{ix}=e^{y}(\cos x+i\sin x)=\underbrace{e^{y}\cos x}_{u}+i\underbrace{e^{y}\sin x}_{v}$ is nowhere analytic since

$$u_x=v_y \Rightarrow -e^{y}\sin x=e^{y}\sin x \Rightarrow 2e^{y}\sin x=0 \Rightarrow \sin x=0$$

and

$$u_y=-v_x \Rightarrow e^{y}\cos x=-e^{y}\cos x \Rightarrow 2e^{y}\cos x=0 \Rightarrow \cos x=0.$$

More precisely, the roots of the equation $\sin x=0$ are $n\pi$ $(n=0,\pm1,\pm2,\ldots)$, and $\cos n\pi=(-1)^n\neq 0$. Consequently, the Cauchy-Riemann equations are not satisfied anywhere.

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$$\frac{d\, g(f(z))}{dz}=g'(f(z))\,f'(z) \qquad (*)$$

if $g(w)$ is entire
and $f(z)$ is entire $\Rightarrow$ they both
analytic in $\mathbb{C}$ (for w and z, respectively)
$\Rightarrow$ $g(f(z))$ is analytic in $\mathbb{C}$ according to $(*)$

$\Rightarrow$ $g(f(z))$ is analytic.

$f=c_1f_1(z)+c_2f_2(z)$, f_1, f_2 entire $\Rightarrow$ differentiable in $\mathbb{C}$ $\Rightarrow$ $c_1f_1+c_2f_2$ is differentiable in $\mathbb{C}$ $\Rightarrow$ $c_1f_1+c_2f_2$ is $\boxed{\text{entire}}$.

4 (a) $f(z)=\dfrac{2z+1}{z(z^2+1)}$ - rational function

$\Rightarrow$ differentiable ~~every~~ except $z(z^2+1)=0$

$\Rightarrow$ $z=0, z=\pm i$ - singular points

5 (Optional)

$g(z) = \sqrt{r}\, e^{i\frac{\theta}{2}} \quad (r > 0, -\pi < \theta < \pi)$

is analytic with $g'(z) = \frac{1}{2g(z)}$

$G(z) = g(2z - 2 + i)$

$\Rightarrow G'(z) = g'(2z-2+i)\,(2z-2+i)'$

$= \frac{1}{2g(2z-2+i)} \cdot 2 = \frac{1}{g(2z-2+i)}$ provided

$|2z-2+i| > 0$

and

$-\pi < \mathrm{Arg}(2z-2+i) < \pi$.

To ensure that $|2z-2+i| > 0$

and $-\pi < \mathrm{Arg}(2z-2+i) < \pi$

we can assume e.g. that $\mathrm{Re}(2z-2+i) > 0$

$\Rightarrow \mathrm{Re}(2z-2+i) = 2x-2 > 0 \Rightarrow x > 1$.

6 $g(z) = \ln r + i\theta \quad (r>0,\ 0<\theta<2\pi)$

$u = \ln r$

$v = \theta$

$$\left.\begin{array}{ll} u_r = \frac{1}{r} & v_r = 0 \\ u_\theta = 0 & v_\theta = 1 \end{array}\right\} \Rightarrow$$ $u_r, v_r, u_\theta, v_\theta$ – exist and continuous for $r>0$, $0<\theta<2\pi$.

$$\left.\begin{array}{l} r u_r = v_\theta = 1 \\ r v_r = -u_\theta = 0 \end{array}\right\} \Rightarrow$$ Cauchy–Riemann eqns. are satisfied

$\Rightarrow$ $g(z)$ is analytic in $r>0$, $0<\theta<2\pi$

and $g'(z) = e^{-i\theta}(u_r + i v_r) = \frac{e^{-i\theta}}{r} = \frac{1}{z}$.

$G(z) = g(z^2+1)$

$G'(z) = g'(z^2+1)\cdot(z^2+1)' = \frac{1}{z^2+1}\, 2z \qquad (*)$

If $x>0$ and $y>0 \Rightarrow \operatorname{Im}(z^2+1)>0$

$\Rightarrow |z^2+1|>0$ and $-\pi < \operatorname{Arg}(z^2+1) < \pi$

$\Rightarrow G(z)$ is analytic.