math 313 Hw05 Solutions



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(a) It is straightforward to show that $u_{xx}+u_{yy}=0$ when $u(x,y)=2x(1-y)$. To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y)=2-2y$. Now

$$u_x=v_y \Rightarrow v_y=2-2y \Rightarrow v(x,y)=2y-y^2+\phi(x).$$

Then

$$u_y=-v_x \Rightarrow -2x=-\phi'(x) \Rightarrow \phi'(x)=2x \Rightarrow \phi(x)=x^2+c.$$

Consequently,

$$v(x,y)=2y-y^2+(x^2+c)=x^2-y^2+2y+c.$$

(d) It is straightforward to show that $u_{xx}+u_{yy}=0$ when $u(x,y)=\frac{y}{x^2+y^2}$. To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y)=-\frac{2xy}{(x^2+y^2)^2}$. Now

$$u_x=v_y \Rightarrow v_y=-\frac{2xy}{(x^2+y^2)^2} \Rightarrow v(x,y)=\frac{x}{x^2+y^2}+\phi(x).$$

Then

$$u_y=-v_x \Rightarrow \frac{x^2-y^2}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}-\phi'(x) \Rightarrow \phi'(x)=0 \Rightarrow \phi(x)=c.$$

Consequently,

$$v(x,y)=\frac{x}{x^2+y^2}+c.$$

**2.** Suppose that $v$ and $V$ are harmonic conjugates of $u$ in a domain $D$. This means that

$$u_x=v_y,\ u_y=-v_x \text{ and } u_x=V_y,\ u_y=-V_x.$$

If $w=v-V$, then,

$$w_x=v_x-V_x=-u_y+u_y=0 \text{ and } w_y=v_y-V_y=u_x-u_x=0.$$

Hence $w(x,y)=c$, where $c$ is a (real) constant (compare the proof of the theorem in Sec. 24). That is, $v(x,y)-V(x,y)=c$.

5. The Cauchy-Riemann equations in polar coordinates are

$$ru_r = v_\theta \text{ and } u_\theta = -rv_r.$$

Now

$$ru_r = v_\theta \Rightarrow ru_{rr} + u_r = v_{\theta r}$$

and

$$u_\theta = -rv_r \Rightarrow u_{\theta\theta} = -rv_{r\theta}.$$

Thus

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = rv_{\theta r} - rv_{r\theta};$$

and, since $v_{\theta r} = v_{r\theta}$, we have

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = 0,$$

which is the polar form of Laplace's equation. To show that $v$ satisfies the same equation, we observe that

$$u_\theta = -rv_r \Rightarrow v_r = -\frac{1}{r}u_\theta \Rightarrow v_{rr} = \frac{1}{r^2}u_\theta - \frac{1}{r}u_{\theta r}$$

and

$$ru_r = v_\theta \Rightarrow v_{\theta\theta} = ru_{r\theta}.$$

Since $u_{\theta r} = u_{r\theta}$, then,

$$r^2 v_{rr} + rv_r + v_{\theta\theta} = u_\theta - ru_{\theta r} - u_\theta + ru_{r\theta} = 0.$$

6. If $u(r,\theta) = \ln r$, then

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = r^2\left(-\frac{1}{r^2}\right) + r\left(\frac{1}{r}\right) + 0 = 0.$$

This tells us that the function $u = \ln r$ is harmonic in the domain $r > 0, 0 < \theta < 2\pi$. Now it follows from the Cauchy-Riemann equation $ru_r = v_\theta$ and the derivative $u_r = \frac{1}{r}$ that $v_\theta = 1$; thus $v(r,\theta) = \theta + \phi(r)$, where $\phi(r)$ is at present an arbitrary differentiable function of $r$. The other Cauchy-Riemann equation $u_\theta = -rv_r$ then becomes $0 = -r\phi'(r)$. That is, $\phi'(r) = 0$; and we see that $\phi(r) = c$, where $c$ is an arbitrary (real) constant. Hence $v(r,\theta) = \theta + c$ is a harmonic conjugate of $u(r,\theta) = \ln r$.

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1. (a) $\exp(2 \pm 3\pi i) = e^2 \exp(\pm 3\pi i) = -e^2$, since $\exp(\pm 3\pi i) = -1$.

(b) $$\exp\frac{2+\pi i}{4} = \left(\exp\frac{1}{2}\right)\left(\exp\frac{\pi i}{4}\right) = \sqrt{e}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= \sqrt{e}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{e}{2}}(1+i).$$

2. $f(z) = 2z^2 - 3 - z e^z + e^{-z}$ is differentiable $\forall z$ $\Rightarrow$ it's is entire.

3. First write

$$\exp(\bar{z}) = \exp(x - iy) = e^x e^{-iy} = e^x \cos y - ie^x \sin y,$$

where $z = x + iy$. This tells us that $\exp(\bar{z}) = u(x,y) + iv(x,y)$, where

$$u(x,y) = e^x \cos y \quad \text{and} \quad v(x,y) = -e^x \sin y.$$

Suppose that the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at some point $z = x + iy$. It is easy to see that, for the functions $u$ and $v$ here, these equations become cos $y = 0$ and sin $y = 0$. But there is no value of $y$ satisfying this pair of equations. We may conclude that, since the Cauchy-Riemann equations fail to be satisfied anywhere, the function $\exp(\bar{z})$ is not analytic anywhere.

5. We first write

$$|\exp(2z+i)| = |\exp[2x + i(2y+1)]| = e^{2x}$$

and

$$|\exp(iz^2)| = |\exp[-2xy + i(x^2 - y^2)]| = e^{-2xy}.$$

Then, since

$$|\exp(2z+i) + \exp(iz^2)| \le |\exp(2z+i)| + |\exp(iz^2)|,$$

it follows that

$$|\exp(2z+i) + \exp(iz^2)| \le e^{2x} + e^{-2xy}.$$

7 To prove that $|\exp(-2z)|<1 \Leftrightarrow \operatorname{Re} z>0$, write

$$|\exp(-2z)| = |\exp(-2x - i2y)| = \exp(-2x).$$

It is then clear that the statement to be proved is the same as $\exp(-2x)<1 \Leftrightarrow x>0$, which is obvious from the graph of the exponential function in calculus.

8. (a) Write $e^z = -2$ as $e^x e^{iy} = 2e^{i\pi}$. This tells us that

$$e^x = 2 \quad \text{and} \quad y = \pi + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

That is,

$$x = \ln 2 \quad \text{and} \quad y = (2n+1)\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Hence

$$z = \ln 2 + (2n+1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

(b) Write $e^z = 1 + \sqrt{3}i$ as $e^x e^{iy} = 2e^{i(\pi/3)}$, from which we see that

$$e^x = 2 \quad \text{and} \quad y = \frac{\pi}{3} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

That is,

$$x = \ln 2 \quad \text{and} \quad y = \left(2n + \frac{1}{3}\right)\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Consequently,

$$z = \ln 2 + \left(2n + \frac{1}{3}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

**10.** *(a)* Suppose that $e^z$ is real. Since $e^z = e^x \cos y + ie^x \sin y$, this means that $e^x \sin y = 0$. Moreover, since $e^x$ is never zero, $\sin y = 0$. Consequently, $y = n\pi \, (n = 0, \pm 1, \pm 2, \ldots)$; that is, $\operatorname{Im} z = n\pi \, (n = 0, \pm 1, \pm 2, \ldots)$.

*(b)* On the other hand, suppose that $e^z$ is pure imaginary. It follows that $\cos y = 0$, or that $y = \frac{\pi}{2} + n\pi \, (n = 0, \pm 1, \pm 2, \ldots)$. That is, $\operatorname{Im} z = \frac{\pi}{2} + n\pi \, (n = 0, \pm 1, \pm 2, \ldots)$.

**14.** The problem here is to establish the identity

$$(\exp z)^n = \exp(nz) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

*(a)* To show that it is true when $n = 0, 1, 2, \ldots$, we use mathematical induction. It is obviously true when $n = 0$. Suppose that it is true when $n = m$, where $m$ is any nonnegative integer. Then

$$(\exp z)^{m+1} = (\exp z)^m (\exp z) = \exp(mz) \exp z = \exp(mz + z) = \exp[(m+1)z].$$

*(b)* Suppose now that $n$ is a negative integer $(n = -1, -2, \ldots)$, and write $m = -n = 1, 2, \ldots$. In view of part *(a)*,

$$(\exp z)^n = \left( \frac{1}{\exp z} \right)^m = \frac{1}{(\exp z)^m} = \frac{1}{\exp(mz)} = \frac{1}{\exp(-nz)} = \exp(nz).$$

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1a

$$\mathrm{Log}(-ei) = \ln|-ei| + i\,\mathrm{Arg}(-ei)$$
$$= \ln e - \frac{i\pi}{2} = 1 - i\frac{\pi}{2}$$

2b

$$\log i = \ln 1 + i\left(\frac{\pi}{2} + 2\pi n\right) = \left(2n + \frac{1}{2}\right)\pi i$$
$$n = 0, \pm 1, \pm 2, \ldots$$

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(b) On the other hand,

$$\mathrm{Log}(-1+i)^2 = \mathrm{Log}(-2i) = \ln 2 - \frac{\pi}{2}i$$

and

$$2\mathrm{Log}(-1+i) = 2\left(\ln\sqrt{2} + i\frac{3\pi}{4}\right) = \ln 2 + \frac{3\pi}{2}i.$$

Hence

$$\mathrm{Log}(-1+i)^2 \neq 2\mathrm{Log}(-1+i).$$

4. (a) Consider the branch

$$\log z = \ln r + i\theta \qquad \left(r > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4}\right).$$

Since

$$\log(i^2) = \log(-1) = \ln 1 + i\pi = \pi i \quad \text{and} \quad 2\log i = 2\left(\ln 1 + i\frac{\pi}{2}\right) = \pi i,$$

we find that $\log(i^2) = 2\log i$ when this branch of $\log z$ is taken.

(7) $\log z = i\frac{\pi}{2}$ (5)

$\Rightarrow z = re^{i\theta}, \ r = 1$

$i\theta + 2\pi n = i\frac{\pi}{2}$

$\theta = \frac{\pi}{2} - 2\pi n$

$z = e^{i\frac{\pi}{2} - 2\pi i n} = i$

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**1.** Suppose that $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$. Then

$$z_1 = r_1 \exp i\Theta_1 \quad \text{and} \quad z_2 = r_2 \exp i\Theta_2,$$

where

$$-\frac{\pi}{2} < \Theta_1 < \frac{\pi}{2} \quad \text{and} \quad -\frac{\pi}{2} < \Theta_2 < \frac{\pi}{2}.$$

The fact that $-\pi < \Theta_1 + \Theta_2 < \pi$ enables us to write

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}[(r_1 r_2)\exp i(\Theta_1 + \Theta_2)] = \ln(r_1 r_2) + i(\Theta_1 + \Theta_2)$$

$$= (\ln r_1 + i\Theta_1) + (\ln r_2 + i\Theta_2) = \operatorname{Log}(r_1 \exp i\Theta_1) + \operatorname{Log}(r_2 \exp i\Theta_2)$$

$$= \operatorname{Log} z_1 + \operatorname{Log} z_2.$$

(6)

(2)

$-\pi < \operatorname{Log} z_{1,2} \leq \pi$

$\Rightarrow$

$-\pi < \operatorname{Log} z_1 + \operatorname{Log} z_2 < 2\pi$

$-\pi < \operatorname{Log}(z_1 z_2) < \pi$

$\operatorname{Log} z_1 + \operatorname{Log} z_2 = \operatorname{Log} z_1 z_2 + 2\pi i n, \quad n = 0, \pm 1, \pm 2, \ldots$

$\Rightarrow n = 0, \pm 1$

(4) $\operatorname{Log} e^{\frac{2}{3}\pi i} = \frac{2}{3}\pi i$

$\operatorname{Log} e^{-\frac{2}{3}\pi i} = -\frac{2}{3}\pi i$

$\operatorname{Log} \dfrac{e^{\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}} = \frac{4}{3}\pi i - 2\pi i = -\frac{2}{3}\pi i$

while

$\operatorname{Log} e^{\frac{2}{3}\pi i} - \operatorname{Log} e^{-\frac{2}{3}\pi i} = -\frac{4}{3}\pi i \neq \operatorname{Log} \dfrac{e^{\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}}$

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p. 104

**1.** In each part below, $n = 0, \pm 1, \pm 2, \ldots$.

(a) $$(1+i)^i = \exp[i\log(1+i)] = \exp\left\{i\left[\ln\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right)\right]\right\}$$

$$= \exp\left[\frac{i}{2}\ln 2 - \left(\frac{\pi}{4} + 2n\pi\right)\right] = \exp\left(-\frac{\pi}{4} - 2n\pi\right)\exp\left(\frac{i}{2}\ln 2\right).$$

Since $n$ takes on all integral values, the term $-2n\pi$ here can be replaced by $+2n\pi$. Thus

$$(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(\frac{i}{2}\ln 2\right).$$

(b) $$(-1)^{1/\pi} = \exp\left[\frac{1}{\pi}\log(-1)\right] = \exp\left\{\frac{1}{\pi}[\ln 1 + i(\pi + 2n\pi)]\right\} = \exp[(2n+1)i].$$

**2.** (a) P.V. $i^i = \exp(i\,\mathrm{Log}\,i) = \exp\left[i\left(\ln 1 + i\frac{\pi}{2}\right)\right] = \exp\left(-\frac{\pi}{2}\right).$

4a

$$(-1+\sqrt{3}i)^{3/2} = [(-1+\sqrt{3}i)^{1/2}]^3$$

$$-1+\sqrt{3}i = 2e^{\frac{2\pi}{3}i}$$

$$\Rightarrow (-1+\sqrt{3}i)^{1/2} = \pm\sqrt{2}\,e^{\frac{\pi i}{3}}$$

$$\Rightarrow [(-1+\sqrt{3}i)^{1/2}]^3 = \pm(\sqrt{2})^3 e^{\pi i}$$

$$= \pm 2\sqrt{2}$$

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$$\cancel{e^{f(z)}} \quad (c^{f(z)})' = (e^{f(z)\log c})' = f'(z)\log c\; e^{f(z)}$$

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8

1. The desired derivatives can be found by writing

$$\frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz}-e^{-iz}}{2i}\right) = \frac{1}{2i}\left(\frac{d}{dz}e^{iz} - \frac{d}{dz}e^{-iz}\right)$$
$$= \frac{1}{2i}\left(ie^{iz} + ie^{-iz}\right) = \frac{e^{iz}+e^{-iz}}{2} = \cos z$$

and

$$\frac{d}{dz}\cos z = \frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right) = \frac{1}{2}\left(\frac{d}{dz}e^{iz} + \frac{d}{dz}e^{-iz}\right)$$
$$= \frac{1}{2}\left(ie^{iz} - ie^{-iz}\right)\cdot\frac{i}{i} = -\frac{e^{iz}-e^{-iz}}{2i} = -\sin z.$$

2a

$$e^{iz_1}e^{iz_2} = (\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2)$$

$$= \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$
$$+ i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)$$

But recall $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$

$$\Rightarrow e^{-iz_1}e^{-iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$
$$- i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)$$

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3. We know from Exercise 2(*b*) that

$$\sin(z+z_2)=\sin z\cos z_2+\cos z\sin z_2.$$

Differentiating each side yields

$$\cos(z+z_2)=\cos z\cos z_2-\sin z\sin z_2.$$

Then, by setting $z=z_1$, we have

$$\cos(z_1+z_2)=\cos z_1\cos z_2-\sin z_1\sin z_2.$$

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$$\frac{d}{dz}\tan z=\frac{d}{dz}\frac{\sin z}{\cos z}=\frac{\cos^2 z+\sin^2 z}{\cos^2 z}=\frac{1}{\cos^2 z}$$

$$\frac{d}{dz}\sec z=\frac{d}{dz}\frac{1}{\cos z}=\frac{\sin z}{\cos^2 z}=\sec z\tan z.$$

(10)

11. By writing $f(z)=\sin\bar{z}=\sin(x-iy)=\sin x\cosh y-i\cos x\sinh y$, we have

$$f(z)=u(x,y)+iv(x,y),$$

where

$$u(x,y)=\sin x\cosh y \quad\text{and}\quad v(x,y)=-\cos x\sinh y.$$

If the Cauchy-Riemann equations $u_x=v_y$, $u_y=-v_x$ are to hold, it is easy to see that

$$\cos x\cosh y=0 \quad\text{and}\quad \sin x\sinh y=0.$$

Since $\cosh y$ is never zero, it follows from the first of these equations that $\cos x=0$; that is, $x=\frac{\pi}{2}+n\pi$ $(n=0\pm1,\pm2,\ldots)$. Furthermore, since $\sin x$ is nonzero for each of these values of $x$, the second equation tells us that $\sinh y=0$, or $y=0$. Thus the Cauchy-Riemann equations hold only at the points

$$z=\frac{\pi}{2}+n\pi \qquad (n=0\pm1,\pm2,\ldots).$$

Evidently, then, there is no neighborhood of any point throughout which $f$ is analytic, and we may conclude that $\sin\bar{z}$ is not analytic anywhere.

The function $f(z)=\cos\bar{z}=\cos(x-iy)=\cos x\cosh y+i\sin x\sinh y$ can be written as

$$f(z)=u(x,y)+iv(x,y),$$

where

$$u(x,y)=\cos x\cosh y \quad\text{and}\quad v(x,y)=\sin x\sinh y.$$

If the Cauchy-Riemann equations $u_x=v_y$, $u_y=-v_x$ hold, then

$$\sin x\cosh y=0 \quad\text{and}\quad \cos x\sinh y=0.$$

The first of these equations tells us that $\sin x=0$, or $x=n\pi$ $(n=0,\pm1,\pm2,\ldots)$. Since $\cos n\pi\neq0$, it follows that $\sinh y=0$, or $y=0$. Consequently, the Cauchy-Riemann equations hold only when

$$z=n\pi \qquad (n=0\pm1,\pm2,\ldots).$$

So there is no neighborhood throughout which $f$ is analytic, and this means that $\cos\bar{z}$ is nowhere analytic.

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**1.** To find the derivatives of sinh*z* and cosh*z*, we write

$$\frac{d}{dz}\sinh z = \frac{d}{dz}\left(\frac{e^z - e^{-z}}{2}\right) = \frac{1}{2}\frac{d}{dz}(e^z - e^{-z}) = \frac{e^z + e^{-z}}{2} = \cosh z$$

and

$$\frac{d}{dz}\cosh z = \frac{d}{dz}\left(\frac{e^z + e^{-z}}{2}\right) = \frac{1}{2}\frac{d}{dz}(e^z + e^{-z}) = \frac{e^z - e^{-z}}{2} = \sinh z.$$

**3.** Identity (9), Sec. 34, is $\sin^2 z + \cos^2 z = 1$. Replacing $z$ by $iz$ here and using the identities

$$\sin(iz) = i\sinh z \quad \text{and} \quad \cos(iz) = \cosh z,$$

we find that $i^2 \sinh^2 z + \cosh^2 z = 1$, or

$$\cosh^2 z - \sinh^2 z = 1.$$

Identity (6), Sec. 34, is $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$. Replacing $z_1$ by $iz_1$ and $z_2$ by $iz_2$ here, we have $\cos[i(z_1 + z_2)] = \cos(iz_1)\cos(iz_2) - \sin(iz_1)\sin(iz_2)$. The same identities that were used just above then lead to

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2.$$

**7.** *(a)* Observe that

$$\sinh(z + \pi i) = \frac{e^{z+\pi i} - e^{-(z+\pi i)}}{2} = \frac{e^z e^{\pi i} - e^{-z} e^{-\pi i}}{2} = \frac{-e^z + e^{-z}}{2} = -\frac{e^z - e^{-z}}{2} = -\sinh z.$$

*(b)* Also,

$$\cosh(z + \pi i) = \frac{e^{z+\pi i} + e^{-(z+\pi i)}}{2} = \frac{e^z e^{\pi i} + e^{-z} e^{-\pi i}}{2} = \frac{-e^z - e^{-z}}{2} = -\frac{e^z + e^{-z}}{2} = -\cosh z.$$

*(c)* From parts *(a)* and *(b)*, we find that

$$\tanh(z + \pi i) = \frac{\sinh(z + \pi i)}{\cosh(z + \pi i)} = \frac{-\sinh z}{-\cosh z} = \frac{\sinh z}{\cosh z} = \tanh z.$$