

①

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①

a

$$\tan^{-1}(2i) = \frac{i}{2} \log \frac{i+2i}{i-2i}$$

$$= \frac{i}{2} \log \frac{3i}{-i} = \frac{i}{2} \log(3)$$

$$= \frac{i}{2} [\ln 3 - i\pi + 2\pi i n]$$

$$= (n + \frac{1}{2})\pi i + \frac{i}{2} \ln 3, \quad n = 0, \pm 1, \dots$$

$$\textcircled{c} \quad \cosh^{-1}(-1) = \log[-1 + (1-1)^{1/2}]$$

$$= (2n+1)\pi i, \quad n = 0, \pm 1, \dots$$

②

a

$$\sinh z = 2$$

$$z = x + iy$$

$$\sinh(x+iy) = \sinh x \cosh iy + i \cosh x \sinh iy$$

$$= \sinh x \cosh y + i \cosh x \sinh y = 2$$

$$\Rightarrow \sinh x = 0 \Rightarrow x = \frac{\pi n + \pi}{2}, \quad n = 0, \pm 1, \dots$$

(2)

$$\sin \times \cos h_2 = 2$$

$$(-1)^n$$

$$\Rightarrow \cos h_2 = (-1)^n 2 \quad \text{but } \cos h_2 > 0 \forall$$

$$\Rightarrow n = 2m$$

$$\cos h_2 = 2$$

$$e^y + e^{-y} = 4$$

$$e^{2y} - 4e^y + 1 = 0$$

$$e^y = 2 \pm \sqrt{3}$$

$$y = \ln(2 \pm \sqrt{3}) = \frac{\pm \ln(2 + \sqrt{3})}{\ln(2 - \sqrt{3})} = \ln\left(\frac{2 - \sqrt{3}}{2 + \sqrt{3}}\right) = -\ln(2 + \sqrt{3})$$

$$\Rightarrow z = \left(2m + \frac{1}{2}\right)\pi \pm i \ln(2 + \sqrt{3})$$

$$m = 0, \pm 1, \pm 2, \dots$$

(6)

$$\sin^{-1} z = -i \log[iz + (1-y)^{1/2}]$$

$$= -i \log(2i \pm i\sqrt{3}) = -i \left[\pm \ln(2 + \sqrt{3}) + \frac{\pi}{2} + 2\pi m \right]$$

$$= \left(\frac{1}{2} + 2m\right)\pi \pm i \ln(2 + \sqrt{3}), \quad m = 0, \pm 1, \pm 2, \dots$$

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① (a)

$$w(t) = u(t) + i v(t)$$

$$\frac{d}{dt} w(t) = \frac{d}{dt} u(t) + i \frac{d}{dt} v(t)$$

$$= -u'(-t) + i v'(-t) = -w'(-t)$$

$$\textcircled{b} \quad \frac{d}{dt} [w(t)]^2 = \left(u^2(t) - v^2(t) + 2i u(t) v(t) \right)'$$

$$= 2u u' - 2v v' + 2i(u'v + v'u')$$

$$= 2(u + iv)(u' + iv') = 2w w'$$

$$\textcircled{2} \textcircled{a) \int_1^2 \left(\frac{1}{t} - i \right)^2 dt = \int_1^2 \left(\frac{1}{t^2} - 1 \right) dt - 2i \int_1^2 \frac{dt}{t} = -\frac{1}{2} - 2i \ln 2 = -\frac{1}{2} - i \ln 4;$$

(c) Since $|e^{-bz}| = e^{-b \operatorname{Re} z}$, we find that

$$\int_0^{\infty} e^{-zt} dt = \lim_{b \rightarrow \infty} \int_0^b e^{-zt} dt = \lim_{b \rightarrow \infty} \left[\frac{e^{-zt}}{-z} \right]_{t=0}^{t=b} = \frac{1}{z} \lim_{b \rightarrow \infty} (1 - e^{-bz}) = \frac{1}{z} \quad \text{when } \operatorname{Re} z > 0.$$

(3.) The problem here is to verify that

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}$$

To do this, we write

$$I = \int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \int_0^{2\pi} e^{i(m-n)\theta} d\theta$$

and observe that when $m \neq n$,

$$I = \left[\frac{e^{i(m-n)\theta}}{i(m-n)} \right]_0^{2\pi} = \frac{1}{i(m-n)} - \frac{1}{i(m-n)} = 0.$$

When $m = n$, I becomes

$$I = \int_0^{2\pi} d\theta = 2\pi;$$

and the verification is complete.

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1. (a) Start by writing

$$I = \int_{-b}^{-a} w(-t) dt = \int_{-b}^{-a} u(-t) dt + i \int_{-b}^{-a} v(-t) dt.$$

The substitution $\tau = -t$ in each of these two integrals on the right then yields

$$I = - \int_b^a u(\tau) d\tau - i \int_b^a v(\tau) d\tau = \int_a^b u(\tau) d\tau + i \int_a^b v(\tau) d\tau = \int_a^b w(\tau) d\tau.$$

That is,

$$\int_{-b}^{-a} w(-t) dt = \int_a^b w(\tau) d\tau.$$

(b) Start with

$$I = \int_a^b w(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt$$

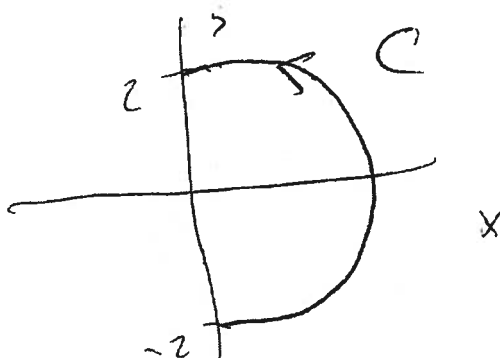
and then make the substitution $t = \phi(\tau)$ in each of the integrals on the right. The result is

$$I = \int_a^\beta u[\phi(\tau)] \phi'(\tau) d\tau + i \int_a^\beta v[\phi(\tau)] \phi'(\tau) d\tau = \int_a^\beta w[\phi(\tau)] \phi'(\tau) d\tau.$$

That is,

$$\int_a^b w(t) dt = \int_a^\beta w[\phi(\tau)] \phi'(\tau) d\tau.$$

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$$(a) z = re^{i\theta} = 2e^{i\theta}$$

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

②

$$(6) \quad z = z(y) = \sqrt{4-y^2} + iy, \quad -2 \leq y \leq 2$$

$$z(\phi(y)) = z e^{i\theta} = z e^{i \arctan \frac{y}{\sqrt{4-y^2}}}$$

$$\theta = \arctan \frac{y}{\sqrt{4-y^2}}$$

$$\cos y = \frac{1}{\sqrt{\tan^2 y + 1}}$$

$$\sin y = \frac{\tan y}{\sqrt{\tan^2 y + 1}}$$

$$\Rightarrow z e^{i \arctan \frac{y}{\sqrt{4-y^2}}} = z \left[\frac{1}{\sqrt{\frac{y^2}{4-y^2} + 1}} + i \frac{\frac{y}{\sqrt{4-y^2}}}{\sqrt{\frac{y^2}{4-y^2} + 1}} \right]$$

$$= 2 \left[\frac{\sqrt{4-y^2}}{2} + \frac{iy}{2} \right] = \sqrt{4-y^2} + iy$$

$$-2 \leq y \leq 2$$

$$\psi'(y) = \left(\arctan \frac{y}{\sqrt{4-y^2}} \right)' = \frac{1}{1 + \frac{y^2}{4-y^2}} \cdot \frac{1 \cdot \sqrt{4-y^2} - \frac{y(-y)}{\sqrt{4-y^2}}}{4-y^2}$$

$$= \frac{1}{4} \left(\frac{4-y^2 + y^2}{\sqrt{4-y^2}} \right) = \frac{1}{4} \frac{4}{\sqrt{4-y^2}} > 0$$

(3)

5. If $w(t) = f[z(t)]$ and $f(z) = u(x, y) + iv(x, y)$, $z(t) = x(t) + iy(t)$, we have

$$w(t) = u[x(t), y(t)] + i[v(x(t), y(t))].$$

The chain rule tells us that

$$\frac{du}{dt} = u_x x' + u_y y' \quad \text{and} \quad \frac{dv}{dt} = v_x x' + v_y y',$$

and so

$$w'(t) = (u_x x' + u_y y') + i(v_x x' + v_y y').$$

In view of the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$, then,

$$w'(t) = (u_x x' - v_x y') + i(v_x x' + u_x y') = (u_x + iv_x)(x' + iy').$$

That is,

$$w'(t) = \{u_x[x(t), y(t)] + iv_x[x(t), y(t)]\}[x'(t) + iy'(t)] = f'[z(t)]z'(t)$$

when $t = t_0$.

(6)

$$0 \leq x \leq 1$$

$$y(x) = \begin{cases} x^3 \sin \frac{\pi}{x} & , 0 < x \leq 1 \\ 0 & , x = 0 \end{cases}$$

$$(a) \quad z = x + iy(x)$$

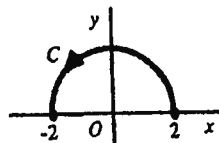
$$y(x) = 0 \Rightarrow x^3 \sin \frac{\pi}{x} = 0$$

$$\Rightarrow \frac{\pi}{x} = \pi n, \quad n = 1, 2, \dots$$

$$\Rightarrow x = \frac{1}{n}, \quad n = 1, 2, \dots$$

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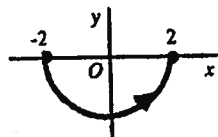
1. (a) Let C be the semicircle $z = 2e^{i\theta}$ ($0 \leq \theta \leq \pi$), shown below.



Then

$$\begin{aligned} \int_C \frac{z+2}{z} dz &= \int_C \left(1 + \frac{2}{z}\right) dz = \int_0^\pi \left(1 + \frac{2}{2e^{i\theta}}\right) 2ie^{i\theta} d\theta = 2i \int_0^\pi (e^{i\theta} + 1) d\theta \\ &= 2i \left[\frac{e^{i\theta}}{i} + \theta \right]_0^\pi = 2i(i + \pi + i) = -4 + 2\pi i. \end{aligned}$$

- (b) Now let C be the semicircle $z = 2e^{i\theta}$ ($\pi \leq \theta \leq 2\pi$) just below.



This is the same as part (a), except for the limits of integration. Thus

$$\int_C \frac{z+2}{z} dz = 2i \left[\frac{e^{i\theta}}{i} + \theta \right]_\pi^{2\pi} = 2i(-i + 2\pi - i - \pi) = 4 + 2\pi i.$$

- (c) Finally, let C denote the entire circle $z = 2e^{i\theta}$ ($0 \leq \theta \leq 2\pi$). In this case,

$$\int_C \frac{z+2}{z} dz = 4\pi i,$$

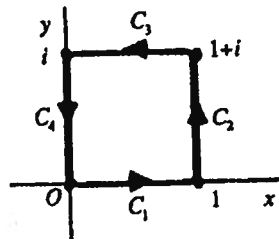
the value here being the sum of the values of the integrals in parts (a) and (b).

2. (a) The arc is $C: z = 1 + e^{i\theta}$ ($\pi \leq \theta \leq 2\pi$). Then

$$\begin{aligned} \int_C (z-1) dz &= \int_\pi^{2\pi} (1 + e^{i\theta} - 1) i e^{i\theta} d\theta = i \int_\pi^{2\pi} e^{i2\theta} d\theta = i \left[\frac{e^{i2\theta}}{2i} \right]_\pi^{2\pi} \\ &= \frac{1}{2} (e^{i4\pi} - e^{i2\pi}) = \frac{1}{2} (1 - 1) = 0. \end{aligned}$$

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3. In this problem, the path C is the sum of the paths C_1 , C_2 , C_3 , and C_4 that are shown below.



The function to be integrated around the closed path C is $f(z) = \pi e^{\pi \bar{z}}$. We observe that $C = C_1 + C_2 + C_3 + C_4$ and find the values of the integrals along the individual legs of the square C .

- (i) Since C_1 is $z = x$ ($0 \leq x \leq 1$),

$$\int_{C_1} \pi e^{\pi \bar{z}} dz = \pi \int_0^1 e^{\pi x} dx = e^{\pi} - 1.$$

- (ii) Since C_2 is $z = 1 + iy$ ($0 \leq y \leq 1$),

$$\int_{C_2} \pi e^{\pi \bar{z}} dz = \pi \int_0^1 e^{\pi(1-iy)} i dy = e^{\pi} \pi i \int_0^1 e^{-i\pi y} dy = 2e^{\pi}.$$

- (iii) Since C_3 is $z = (1-x) + i$ ($0 \leq x \leq 1$),

$$\int_{C_3} \pi e^{\pi \bar{z}} dz = \pi \int_0^1 e^{\pi((1-x)-i)} (-1) dx = \pi e^{\pi} \int_0^1 e^{-\pi x} dx = e^{\pi} - 1.$$

- (iv) Since C_4 is $z = i(1-y)$ ($0 \leq y \leq 1$),

$$\int_{C_4} \pi e^{\pi \bar{z}} dz = \pi \int_0^1 e^{-\pi(1-y)i} (-i) dy = \pi i \int_0^1 e^{i\pi y} dy = -2.$$

Finally, then, since

$$\int_C \pi e^{\pi \bar{z}} dz = \int_{C_1} \pi e^{\pi \bar{z}} dz + \int_{C_2} \pi e^{\pi \bar{z}} dz + \int_{C_3} \pi e^{\pi \bar{z}} dz + \int_{C_4} \pi e^{\pi \bar{z}} dz,$$

we find that

$$\int_C \pi e^{\pi \bar{z}} dz = 4(e^{\pi} - 1).$$

4. The path C is the sum of the paths

$$C_1: z = x + ix^3 \quad (-1 \leq x \leq 0) \quad \text{and} \quad C_2: z = x + ix^3 \quad (0 \leq x \leq 1).$$

Using

$$f(z) = 1 \text{ on } C_1 \quad \text{and} \quad f(z) = 4y = 4x^3 \text{ on } C_2,$$

we have

$$\begin{aligned} \int_C f(z) dz &= \int_{C_1} f(z) dz + \int_{C_2} f(z) dz = \int_{-1}^0 1(1 + i3x^2) dx + \int_0^1 4x^3(1 + i3x^2) dx \\ &= \int_{-1}^0 dx + 3i \int_{-1}^0 x^2 dx + 4 \int_0^1 x^3 dx + 12i \int_0^1 x^5 dx \\ &= [x]_{-1}^0 + i[x^3]_{-1}^0 + [x^4]_0^1 + 2i[x^6]_0^1 = 1 + i + 1 + 2i = 2 + 3i. \end{aligned}$$

8. Let C be the positively oriented circle $|z| = 1$, with parametric representation $z = e^{i\theta}$ ($0 \leq \theta \leq 2\pi$), and let m and n be integers. Then

$$\int_C z^m \bar{z}^n dz = \int_0^{2\pi} (e^{i\theta})^m (e^{-i\theta})^n i e^{i\theta} d\theta = i \int_0^{2\pi} e^{i(m+1)\theta} e^{-in\theta} d\theta.$$

But we know from Exercise 3, Sec. 38, that

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}$$

Consequently,

$$\int_C z^m \bar{z}^n dz = \begin{cases} 0 & \text{when } m+1 \neq n, \\ 2\pi i & \text{when } m+1 = n. \end{cases}$$

11. (a) The function $f(z)$ is continuous on a smooth arc C , which has a parametric representation $z = z(t)$ ($a \leq t \leq b$). Exercise 1(b), Sec. 38, enables us to write

$$\int_a^b f[z(t)] z'(t) dt = \int_a^b f[Z(\tau)] z'[\phi(\tau)] \phi'(\tau) d\tau,$$

where

$$Z(\tau) = z[\phi(\tau)]$$

$$(\alpha \leq \tau \leq \beta).$$

But expression (14), Sec. 38, tells us that

$$z'[\phi(\tau)] \phi'(\tau) = Z'(\tau);$$

and so

(4)

$$\int_a^b f[z(t)]z'(t)dt = \int_a^b f[Z(\tau)]Z'(\tau)d\tau.$$

- (b) Suppose that C is any contour and that $f(z)$ is piecewise continuous on C . Since C can be broken up into a finite chain of smooth arcs on which $f(z)$ is continuous, the identity obtained in part (a) remains valid.

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6. To integrate the branch

$$z^{-1+i} = e^{(-1+i)\log z}$$

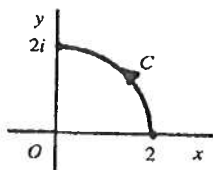
$$(|z| > 0, 0 < \arg z < 2\pi)$$

around the circle $C: z = e^{i\theta}$ ($0 \leq \theta \leq 2\pi$), write

$$\int_C z^{-1+i} dz = \int_C e^{(-1+i)\log z} dz = \int_0^{2\pi} e^{(-1+i)(i\theta)} i e^{i\theta} d\theta = i \int_0^{2\pi} e^{-i\theta-\theta} e^{i\theta} d\theta = i \int_0^{2\pi} e^{-\theta} d\theta = i(1 - e^{-2\pi}).$$

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1. Let C be the arc of the circle $|z| = 2$ shown below.



Without evaluating the integral, let us find an upper bound for $\left| \int_C \frac{dz}{z^2-1} \right|$. To do this, we note that if z is a point on C ,

$$|z^2-1| \geq ||z|^2-1| = |4-1| = 3.$$

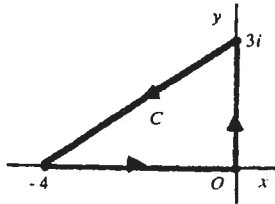
Thus

$$\left| \frac{1}{z^2-1} \right| = \frac{1}{|z^2-1|} \leq \frac{1}{3}.$$

Also, the length of C is $\frac{1}{4}(4\pi) = \pi$. So, taking $M = \frac{1}{3}$ and $L = \pi$, we find that

$$\left| \int_C \frac{dz}{z^2-1} \right| \leq ML = \frac{\pi}{3}.$$

3. The contour C is the closed triangular path shown below.



To find an upper bound for $\left| \int_C (e^z - \bar{z}) dz \right|$, we let z be a point on C and observe that

$$|e^z - \bar{z}| \leq |e^z| + |\bar{z}| = e^x + \sqrt{x^2 + y^2}.$$

But $e^x \leq 1$ since $x \leq 0$, and the distance $\sqrt{x^2 + y^2}$ of the point z from the origin is always less than or equal to 4. Thus $|e^z - \bar{z}| \leq 5$ when z is on C . The length of C is evidently 12. Hence, by writing $M = 5$ and $L = 12$, we have

$$\left| \int_C (e^z - \bar{z}) dz \right| \leq ML = 60.$$

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- (1.) The function z^n ($n = 0, 1, 2, \dots$) has the antiderivative $z^{n+1}/(n+1)$ everywhere in the finite plane. Consequently, for any contour C from a point z_1 to a point z_2 ,

$$\int_C z^n dz = \int_{z_1}^{z_2} z^n dz = \left[\frac{z^{n+1}}{n+1} \right]_{z_1}^{z_2} = \frac{z_2^{n+1}}{n+1} - \frac{z_1^{n+1}}{n+1} = \frac{1}{n+1} (z_2^{n+1} - z_1^{n+1}).$$

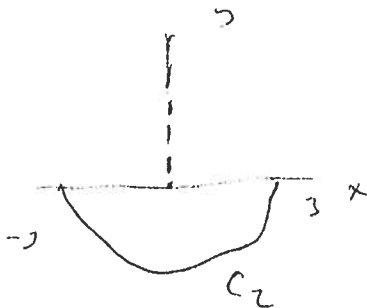
(2.) (a) $\int_1^{1+i} e^{\pi z} dz = \left[\frac{e^{\pi z}}{\pi} \right]_1^{1+i} = \frac{e^{\pi(1+i)} - e^{\pi}}{\pi} = \frac{i+1}{\pi} = \frac{1+i}{\pi}.$

(b) $\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz = 2 \left[\sin\left(\frac{z}{2}\right) \right]_0^{\pi+2i} = 2 \sin\left(\frac{\pi}{2} + i\right) = 2 \frac{e^{i(\frac{\pi}{2}+i)} - e^{-i(\frac{\pi}{2}+i)}}{2i} = -i(e^{i\pi/2}e^{-1} - e^{-i\pi/2}e) = -i\left(\frac{i}{e} + ie\right) = \frac{1}{e} + e = e + \frac{1}{e}.$

(c) $\int_1^3 (z-2)^4 dz = \left[\frac{(z-2)^5}{5} \right]_1^3 = \frac{1}{5} - \frac{1}{5} = 0.$

(4) $f_2(z) = r e^{i\theta/2}$

$r > 0, \quad \frac{\pi}{2} < \theta < \frac{5}{2}\pi$



$$\int_{C_2} z^{1/2} dz$$

$F_2(z) = \frac{2}{3} r \sqrt{r} e^{i\frac{3\theta}{2}}, \quad \frac{\pi}{2} < \theta < \frac{5}{2}\pi$

$$\int_{C_2} z^{1/2} dz = \frac{2}{3} 3\sqrt{3} e^{i\frac{3\pi}{2}} - \frac{2}{3} 3\sqrt{3} e^{i\frac{3\pi}{2}} = 2\sqrt{3} (-1 - (-i)) = 2\sqrt{3} (-1+i)$$

5. Let C denote any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis. This exercise asks us to evaluate the integral

$$I = \int_{-1}^1 z^i dz,$$

where z^i denotes the principal branch

$$z^i = \exp(i \operatorname{Log} z) \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi).$$

An antiderivative of this branch *cannot* be used since the branch is not even defined at $z = -1$. But the integrand can be replaced by the branch

$$z^i = \exp(i \log z) \quad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2} \right)$$

since it agrees with the integrand along C . Using an antiderivative of this new branch, we can now write

$$\begin{aligned} I &= \left[\frac{z^{i+1}}{i+1} \right]_{-1}^1 = \frac{1}{i+1} \left[(1)^{i+1} - (-1)^{i+1} \right] = \frac{1}{i+1} \left[e^{(i+1)\log 1} - e^{(i+1)\log(-1)} \right] \\ &= \frac{1}{i+1} \left[e^{(i+1)(\ln 1 + i0)} - e^{(i+1)(\ln 1 + i\pi)} \right] = \frac{1}{i+1} \left(1 - e^{-\pi} e^{i\pi} \right) = \frac{1+e^{-\pi}}{1+i} \cdot \frac{1-i}{1-i} \\ &= \frac{1+e^{-\pi}}{2} (1-i). \end{aligned}$$