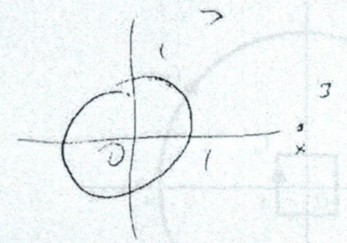
math 313 HW 07 Solutions

(1)

p. 160 (1)

(a)



$f(z) = \dfrac{z^2}{z-3}$ Function is analytic everywhere except $z=3$

$\Rightarrow$ analytic inside and on $|z|=1$ circle

$\Rightarrow$ by Cauchy - Goursat thm

$$\int_C \frac{z^2}{z-3}\, dz = 0$$

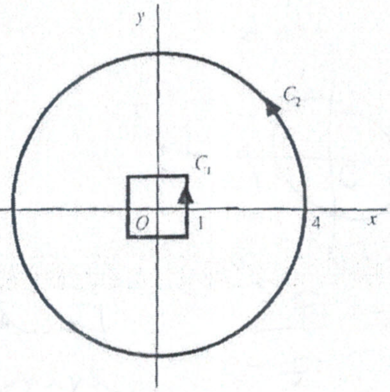
(c) $f(z) = \dfrac{1}{z^2+2z+2} = \dfrac{1}{(z+1+i)(z+1-i)}$

$z = -1 \pm \sqrt{1-2} = -1 \pm i$

$f(z)$ is analytic everywhere except $z = -1 \pm i$ $\Rightarrow$ inside and on C_0 $f(z)$ is analytic

$$\Rightarrow \int_C \frac{dz}{z^2+2z+2} = 0$$

2. The contours C_1 and C_2 are as shown in the figure below.

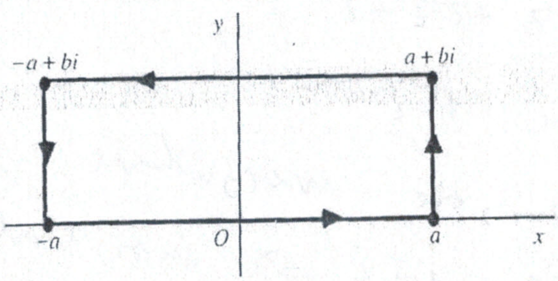


In each of the cases below, the singularities of the integrand lie inside C_1 or outside of C_2; and so the integrand is analytic on the contours and between them. Consequently,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

(b) When $f(z) = \dfrac{z+2}{\sin(z/2)}$, the singularities are at $z = 2n\pi\ (n = 0, \pm 1, \pm 2, \ldots)$.

4. (a) In order to derive the integration formula in question, we integrate the function e^{-z^2} around the closed rectangular path shown below.



Since the lower horizontal leg is represented by $z = x\ (-a \le x \le a)$, the integral of e^{-z^2} along that leg is

$$\int_{-a}^{a} e^{-x^2}\,dx = 2\int_{0}^{a} e^{-x^2}\,dx.$$

Since the opposite direction of the upper horizontal leg has parametric representation $z = x + bi$ $(-a \le x \le a)$, the integral of e^{-z^2} along the upper leg is

$$-\int_{-a}^{a} e^{-(x+bi)^2}\,dx = -e^{b^2}\int_{-a}^{a} e^{-x^2}e^{-i2bx}\,dx = -e^{b^2}\int_{-a}^{a} e^{-x^2}\cos 2bx\,dx + ie^{b^2}\int_{-a}^{a} e^{-x^2}\sin 2bx\,dx,$$

or simply

$$-2e^{b^2}\int_{0}^{a} e^{-x^2}\cos 2bx\,dx.$$

Since the right-hand vertical leg is represented by $z = a + iy$ $(0 \le y \le b)$, the integral of e^{-z^2} along it is

$$\int_{0}^{b} e^{-(a+iy)^2}\,i\,dy = ie^{-a^2}\int_{0}^{b} e^{y^2}e^{-i2ay}\,dy.$$

Finally, since the opposite direction of the left-hand vertical leg has the representation $z = -a + iy$ $(0 \le y \le b)$, the integral of e^{-z^2} along that vertical leg is

$$-\int_{0}^{b} e^{-(-a+iy)^2}\,i\,dy = -ie^{-a^2}\int_{0}^{b} e^{y^2}e^{i2ay}\,dy.$$

According to the Cauchy-Goursat theorem, then,

$$2\int_{0}^{a} e^{-x^2}\,dx - 2e^{b^2}\int_{0}^{a} e^{-x^2}\cos 2bx\,dx + ie^{-a^2}\int_{0}^{b} e^{y^2}e^{-i2ay}\,dy - ie^{-a^2}\int_{0}^{b} e^{y^2}e^{i2ay}\,dy = 0;$$

and this reduces to

$$\int_{0}^{a} e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_{0}^{a} e^{-x^2}\,dx + e^{-(a^2+b^2)}\int_{0}^{b} e^{y^2}\sin 2ay\,dy.$$

(b) We now let $a \to \infty$ in the final equation in part *(a)*, keeping in mind the known integration formula

$$\int_{0}^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

and the fact that

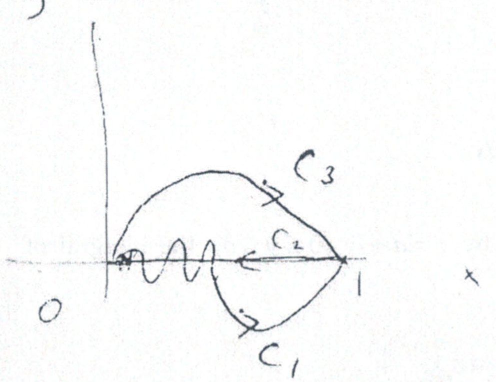
$$\left| e^{-(a^2+b^2)}\int_{0}^{b} e^{y^2}\sin 2ay\,dy \right| \le e^{-(a^2+b^2)}\int_{0}^{b} e^{y^2}\,dy \to 0 \text{ as } a \to \infty.$$

The result is

$$\int_{0}^{\infty} e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2} \qquad (b > 0).$$

5.

$$y(x) = \begin{cases} x^3 \sin\frac{\pi}{x}, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$



C_2 connects 0 and 1

C_1, C_2, C_3 — smooth arcs

$C_1 - C_3$ — simple closed contour
(because $z(x)$ is piecewise continuous)

the same for
$C_2 + C_3$

$$\Rightarrow \int_{C_1} f(z)dz + \int_{-C_3} f(z)dz = 0 \Rightarrow \int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

$$\int_{C_2} f(z)dz + \int_{C_3} f(z)dz = 0$$ by Cauchy-Goursat thm.

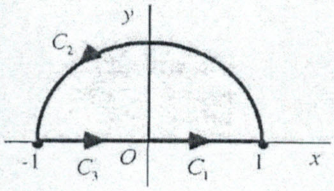
5

$$\int_C f(z)dz = \int_{C_1+C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

$$\int_{C_3} f(z)dz$$

$$= \left(\int_{C_3} + \int_{C_2}\right) f(z)dz = 0$$

6. We let C denote the entire boundary of the semicircular region appearing below. It is made up of the leg C_1 from the origin to the point $z=1$, the semicircular arc C_2 that is shown, and the leg C_3 from $z=-1$ to the origin. Thus $C = C_1 + C_2 + C_3$.



We also let $f(z)$ be a continuous function that is defined on this closed semicircular region by writing $f(0)=0$ and using the branch

$$f(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r>0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the multiple-valued function $z^{1/2}$. The problem here is to evaluate the integral of $f(z)$ around C by evaluating the integrals along the individual paths $C_1, C_2,$ and C_3 and then adding the results. In each case, we write a parametric representation for the path (or a related one) and then use it to evaluate the integral along the particular path.

(i) C_1: $z = re^{i0}$ $(0 \le r \le 1)$. Then

$$\int_{C_1} f(z)dz = \int_0^1 \sqrt{r}\cdot 1\, dr = \left[\frac{2}{3}r^{3/2}\right]_0^1 = \frac{2}{3}.$$

(ii) C_2: $z = 1\cdot e^{i\theta}$ $(0 \le \theta \le \pi)$. Then

$$\int_{C_2} f(z)dz = \int_0^\pi e^{i\theta/2}\cdot ie^{i\theta}d\theta = i\int_0^\pi e^{i3\theta/2}d\theta = i\left[\frac{2}{3i}e^{i3\theta/2}\right]_0^\pi = \frac{2}{3}(-i-1) = -\frac{2}{3}(1+i).$$

(iii) $-C_3$: $z = re^{i\pi}$ $(0 \le r \le 1)$. Then

$$\int_{C_3} f(z)dz = -\int_{-C_3} f(z)dz = -\int_0^1 \sqrt{r}e^{i\pi/2}(-1)dr = i\int_0^1 \sqrt{r}dr = i\left[\frac{2}{3}r^{3/2}\right]_0^1 = \frac{2}{3}i.$$

The desired result is

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz = \frac{2}{3} - \frac{2}{3}(1+i) + \frac{2}{3}i = 0.$$

The Cauchy-Goursat theorem does not apply since $f(z)$ is not analytic at the origin, or even defined on the negative imaginary axis.