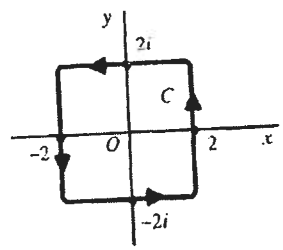
HW 08 Solutions math 313 ①

p. 170

① In this problem, we let C denote the square contour shown in the figure below.



(a) $\int_C \frac{e^{-z}\,dz}{z-(\pi i/2)} = 2\pi i\left[e^{-z}\right]_{z=\pi i/2} = 2\pi i(-i) = 2\pi.$

(b) $\int_C \frac{\cos z}{z(z^2+8)}dz = \int_C \frac{(\cos z)/(z^2+8)}{z-0}dz = 2\pi i\left[\frac{\cos z}{z^2+8}\right]_{z=0} = 2\pi i\left(\frac{1}{8}\right) = \frac{\pi i}{4}.$

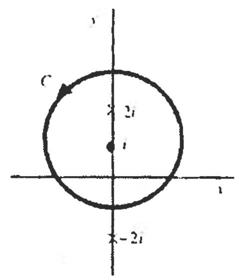
(c) $\int_C \frac{z\,dz}{2z+1} = \int_C \frac{z/2}{z-(-1/2)}dz = 2\pi i\left[\frac{z}{2}\right]_{z=-1/2} = 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi i}{2}.$

(d) $\int_C \frac{\cosh z}{z^4}dz = \int_C \frac{\cosh z}{(z-0)^{3+1}}dz = \frac{2\pi i}{3!}\left[\frac{d^3}{dz^3}\cosh z\right]_{z=0} = \frac{\pi i}{3}(0) = 0.$

(e) $\int_C \frac{\tan(z/2)}{(z-x_0)^2}dz = \int_C \frac{\tan(z/2)}{(z-x_0)^{1+1}}dz = \frac{2\pi i}{1!}\left[\frac{d}{dz}\tan\left(\frac{z}{2}\right)\right]_{z=x_0}$

$= 2\pi i\left(\frac{1}{2}\sec^2\frac{x_0}{2}\right) = i\pi\sec^2\left(\frac{x_0}{2}\right)$ when $-2 < x_0 < 2$.

2. Let C denote the positively oriented circle $|z-i|=2$, shown below.



(a) The Cauchy integral formula enables us to write

$$\int_C \frac{dz}{z^2+4} = \int_C \frac{dz}{(z-2i)(z+2i)} = \int_C \frac{1/(z+2i)}{z-2i}\,dz = 2\pi i\left(\frac{1}{z+2i}\right)_{z=2i} = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}.$$

(4)

$$g(z) = \int_C \frac{s^3+2s}{(s-z)^3}\,ds$$

If z is inside C

$\Rightarrow$

$$g(z) = \int_C \frac{s^3+2s\,ds}{(s-z)^3}$$

$$= -\int_{C_1} \frac{s^3+2s\,ds}{(s-z)^3} = -\frac{2\pi i}{2!}\frac{(z^3+2z)'''}{-1} = \frac{-\pi i\,6}{-1}$$

$$= 6\pi i$$

If z is outside of C

$\Rightarrow$

$$g(z) = \int_C \frac{s^3+2s\,ds}{(s-z)^3} = 0$$

5. Suppose that a function f is analytic inside and on a simple closed contour C and that z_0 is not on C. If z_0 is inside C, then

$$\int_C \frac{f'(z)dz}{z-z_0} = 2\pi i f'(z_0) \quad \text{and} \quad \int_C \frac{f(z)dz}{(z-z_0)^2} = \int_C \frac{f(z)dz}{(z-z_0)^{1+1}} = \frac{2\pi i}{1!} f'(z_0).$$

Thus

$$\int_C \frac{f'(z)dz}{z-z_0} = \int_C \frac{f(z)dz}{(z-z_0)^2}.$$

The Cauchy-Goursat theorem tells us that this last equation is also valid when z_0 is exterior to C, each side of the equation being 0.

7. Let C be the unit circle $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$, and let a denote any real constant. The Cauchy integral formula reveals that

$$\int_C \frac{e^{az}}{z} dz = \int_C \frac{e^{az}}{z-0} dz = 2\pi i \left[e^{az}\right]_{z=0} = 2\pi i.$$

On the other hand, the stated parametric representation for C gives us

$$\int_C \frac{e^{az}}{z} dz = \int_{-\pi}^{\pi} \frac{\exp(ae^{i\theta})}{e^{i\theta}} ie^{i\theta} d\theta = i \int_{-\pi}^{\pi} \exp[a(\cos\theta + i\sin\theta)]d\theta$$

$$= i \int_{-\pi}^{\pi} e^{a\cos\theta} e^{ia\sin\theta} d\theta = i \int_{-\pi}^{\pi} e^{a\cos\theta}[\cos(a\sin\theta) + i\sin(a\sin\theta)]d\theta$$

$$= -\int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)d\theta + i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta.$$

Equating these two different expressions for the integral $\int_C \frac{e^{az}}{z} dz$, we have

$$-\int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)d\theta + i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = 2\pi i.$$

Then, by equating the imaginary parts on each side of this last equation, we see that

$$\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = 2\pi;$$

and, since the integrand here is even,

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = \pi.$$

9

8. (a) The binomial formula enables us to write

(optional)

$$P_n(z) = \frac{1}{n!2^n}\frac{d^n}{dz^n}(z^2-1)^n = \frac{1}{n!2^n}\frac{d^n}{dz^n}\sum_{k=0}^{n}\binom{n}{k}z^{2n-2k}(-1)^k.$$

We note that the highest power of z appearing under the derivative is z^{2n}, and differentiating it n times brings it down to z^n. So $P_n(z)$ is a polynomial of degree n.

(b) We let C denote any positively oriented simple closed contour surrounding a fixed point z. The Cauchy integral formula for derivatives tells us that

$$\frac{d^n}{dz^n}(z^2-1)^n = \frac{n!}{2\pi i}\int_C \frac{(s^2-1)^n}{(s-z)^{n+1}}ds \qquad (n=0,1,2,\ldots).$$

Hence the polynomials $P_n(z)$ in part *(a)* can be written

$$P_n(z) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s-z)^{n+1}}ds \qquad (n=0,1,2,\ldots).$$

(c) Note that

$$\frac{(s^2-1)^n}{(s-1)^{n+1}} = \frac{(s-1)^n(s+1)^n}{(s-1)^{n+1}} = \frac{(s+1)^n}{s-1}.$$

Referring to the final result in part *(b)*, then, we have

$$P_n(1) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s-1)^{n+1}}ds = \frac{1}{2^n}\frac{1}{2\pi i}\int_C \frac{(s+1)^n}{s-1}ds = \frac{1}{2^n}2^n = 1 \qquad (n=0,1,2,\ldots).$$

Also, since

$$\frac{(s^2-1)^n}{(s+1)^{n+1}} = \frac{(s-1)^n(s+1)^n}{(s+1)^{n+1}} = \frac{(s-1)^n}{s+1},$$

we have

$$P_n(-1) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s+1)^{n+1}}ds = \frac{1}{2^n}\frac{1}{2\pi i}\int_C \frac{(s-1)^n}{s+1}ds = \frac{1}{2^n}(-2)^n = (-1)^n \quad (n=0,1,2,\ldots).$$

9. We are asked to show that
(optional)

$$f''(z)=\frac{1}{\pi i}\int_C \frac{f(s)ds}{(s-z)^3}.$$

(a) In view of the expression for $f'(z)$ in the lemma,

$$\begin{aligned}\frac{f'(z+\Delta z)-f'(z)}{\Delta z}&=\frac{1}{2\pi i}\int_C\left[\frac{1}{(s-z-\Delta z)^2}-\frac{1}{(s-z)^2}\right]\frac{f(s)ds}{\Delta z}\\&=\frac{1}{2\pi i}\int_C\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}f(s)ds.\end{aligned}$$

Then

$$\begin{aligned}\frac{f'(z+\Delta z)-f'(z)}{\Delta z}-\frac{1}{\pi i}\int_C\frac{f(s)ds}{(s-z)^3}&=\frac{1}{2\pi i}\int_C\left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}-\frac{2}{(s-z)^3}\right]f(s)ds\\&=\frac{1}{2\pi i}\int_C\frac{3(s-z)\Delta z-2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}f(s)ds.\end{aligned}$$

(b) We must show that

$$\left|\int_C\frac{3(s-z)\Delta z-2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}f(s)ds\right|\le\frac{(3D|\Delta z|+2|\Delta z|^2)M}{(d-|\Delta z|)^2d^3}L.$$

Now D, d, M, and L are as in the statement of the exercise in the text. The triangle inequality tells us that

$$|3(s-z)\Delta z-2(\Delta z)^2|\le 3|s-z||\Delta z|+2|\Delta z|^2\le 3D|\Delta z|+2|\Delta z|^2.$$

Also, we know from the verification of the expression for $f'(z)$ in the lemma that $|s-z-\Delta z|\ge d-|\Delta z|>0$; and this means that

$$|(s-z-\Delta z)^2(s-z)^3|\ge(d-|\Delta z|)^2d^3>0.$$

This gives the desired inequality.

(c) If we let Δz tend to 0 in the inequality obtained in part *(b)* we find that

$$\lim_{\Delta z\to 0}\frac{1}{2\pi i}\int_C\frac{3(s-z)\Delta z-2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}f(s)ds=0.$$

This, together with the result in part *(a)*, yields the desided expression for $f''(z)$.

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1. $u \le u_0 \ \forall (x,y)$

$f = u + iv$ - entire function

Define $g(z) = e^{f(z)}$

$\Rightarrow |g(z)| = |e^{u+iv}| = e^u \le e^{u_0}$

$\Rightarrow$ $g(z)$ is entire and bounded

$\Rightarrow$ by Liouville's thm

$g(z) = \text{const} \Rightarrow f(z) = \text{const}.$

2. $w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots \frac{a_{n-1}}{z}.$

Choose R so large that $\forall_{0 \le j \le n-1}$ $\left|\frac{a_j}{R^{n-j}}\right| < \frac{|a_n|}{n}$

$\Rightarrow$ for $|z| > R$: $\left|\frac{a_j}{z^{n-j}}\right| \le \left|\frac{a_j}{R^{n-j}}\right| < \frac{|a_n|}{n}$

$\Rightarrow |p(z)| = |a_n + w|\,|z|^n \le \left(|a_n| + \left|\frac{a_0}{z^n}\right| + \dots + \left|\frac{a_{n-1}}{z}\right|\right)|z|^n$

$< \left(|a_n| + n\frac{|a_n|}{n}\right)|z|^n = 2|a_n||z|^n$ ∎.

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p. 188

1. Let us use definition (2), Sec. 55, to show that the sequence

$$z_n = -2 + i\frac{(-1)^n}{n^2} \qquad (n = 1, 2, \ldots)$$

converges to -2. Observe that $|z_n - (-2)| = \frac{1}{n^2}$. Thus, for each $\varepsilon > 0$,

$$|z_n - (-2)| < \varepsilon \quad \text{whenever} \quad n > n_0,$$

where n_0 is any positive integer such that $n_0 \geq \frac{1}{\sqrt{\varepsilon}}$.

2. Note that if $z_n = 2 + i\frac{(-1)^n}{n^2}$ $(n = 1, 2, \ldots)$, then

$$\Theta_{2n} = \operatorname{Arg} z_{2n} \to 0 \quad \text{and} \quad \Theta_{2n-1} = \operatorname{Arg} z_{2n-1} \to 0 \qquad (n = 1, 2, \ldots)$$

Hence the sequence Θ_n $(n = 1, 2, \ldots)$ does converge.

3. Suppose that $\lim_{n\to\infty} z_n = z$. That is, for each $\varepsilon > 0$, there is a positive integer n_0 such that $|z_n - z| < \varepsilon$ whenever $n > n_0$. In view of the inequality (see Sec. 4)

$$|z_n - z| \geq ||z_n| - |z||,$$

it follows that $||z_n| - |z|| < \varepsilon$ whenever $n > n_0$. That is, $\lim_{n\to\infty} |z_n| = |z|$.

4. The summation formula found in the example in Sec. 56 can be written

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z} \quad \text{when} \quad |z| < 1.$$

If we put $z = re^{i\theta}$, where $0 < r < 1$, the left-hand side becomes

$$\sum_{n=1}^{\infty}(re^{i\theta})^n = \sum_{n=1}^{\infty} r^n e^{in\theta} = \sum_{n=1}^{\infty} r^n \cos n\theta + i\sum_{n=1}^{\infty} r^n \sin n\theta;$$

and the right-hand side takes the form

$$\frac{re^{i\theta}}{1-re^{i\theta}} \cdot \frac{1-re^{-i\theta}}{1-re^{-i\theta}} = \frac{re^{i\theta}-r^2}{1-r(e^{i\theta}+e^{-i\theta})+r^2} = \frac{r\cos\theta - r^2 + ir\sin\theta}{1-2r\cos\theta + r^2}.$$

Thus

$$\sum_{n=1}^{\infty} r^n \cos n\theta + i\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\cos\theta - r^2}{1-2r\cos\theta + r^2} + i\frac{r\sin\theta}{1-2r\cos\theta + r^2}.$$

Equating the real parts on each side here and then the imaginary parts, we arrive at the summation formulas

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1-2r\cos\theta + r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1-2r\cos\theta + r^2},$$

where $0 < r < 1$. These formulas clearly hold when $r = 0$ too.

6. Suppose that $\sum_{n=1}^{\infty} z_n = S$. To show that $\sum_{n=1}^{\infty} \overline{z}_n = \overline{S}$, we write $z_n = x_n + iy_n$, $S = X + iY$ and appeal to the theorem in Sec. 56. First of all, we note that

$$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y.$$

Then, since $\sum_{n=1}^{\infty}(-y_n) = -Y$, it follows that

$$\sum_{n=1}^{\infty} \overline{z}_n = \sum_{n=1}^{\infty}(x_n - iy_n) = \sum_{n=1}^{\infty}[x_n + i(-y_n)] = X - iY = \overline{S}.$$

9 (optional)

$z_n \to z$ for $n \to \infty$

choose n_0 such that $|z_n - z| < \varepsilon$ for $n > n_0$

(a) $\Rightarrow |z_n| = |z + (z_n - z)| < |z| + 1 = M_0 \quad \forall\, n > n_0$

choose $M = \max(M_0, |z_1|, \ldots |z_{n-1}|)$

(b) $z_n = x_n + i y_n$.

~~\#~~ From convergence of x_n, y_n

$\Rightarrow |x_n| \le M_1, \; |y_n| \le y_2$

$\Rightarrow |z_n| = \sqrt{x_n^2 + y_n^2} \le \sqrt{M_1^2 + M_2^2} = M.$

p. 195

1. Replace z by z^2 in the known series

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad (|z| < \infty)$$

to get

$$\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n}}{(2n)!} \qquad (|z| < \infty).$$

Then, multiplying through this last equation by z, we have the desired result:

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (|z| < \infty).$$

2. (b) Replacing z by $z-1$ in the known expansion

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z| < \infty),$$

we have

$$e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z| < \infty).$$

So

$$e^z = e^{z-1}e = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z| < \infty).$$

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3. We want to find the Maclaurin series for the function

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9} \cdot \frac{1}{1+(z^4/9)}.$$

To do this, we first replace z by $-(z^4/9)$ in the known expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z|<1),$$

as well as its condition of validity, to get

$$\frac{1}{1+(z^4/9)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n}} z^{4n} \qquad (|z|<\sqrt{3}).$$

Then, if we multiply through this last equation by $\frac{z}{9}$, we have the desired expansion:

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+2}} z^{4n+1} \qquad (|z|<\sqrt{3}).$$

9 $f(z) = \sin z$.

(1) $n=0 \Rightarrow f^{(0)}(0) = \sin 0 = 0$,

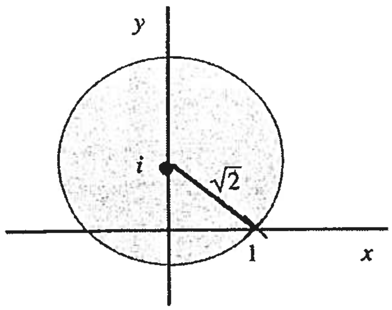
$f'(0) = \cos 0 = 1 = (-1)^0$

(2) ~~$n \to 0$~~ $n > 0$

$(\sin z)^{(2n)} \Big|_{z=0} = (-1)^n \sin 0 = 0$

$(\sin z)^{2n+1} = (-1)^n \cos 0 = (-1)^n$

7. The function $\dfrac{1}{1-z}$ has a singularity at $z=1$. So the Taylor series about $z=i$ is valid when $|z-i|<\sqrt{2}$, as indicated in the figure below.



To find the series, we start by writing

$$\frac{1}{1-z}=\frac{1}{(1-i)-(z-i)}=\frac{1}{1-i}\cdot\frac{1}{1-(z-i)/(1-i)}.$$

This suggests that we replace z by $(z-i)/(1-i)$ in the known expansion

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n \qquad (|z|<1)$$

and then multiply through by $\dfrac{1}{1-i}$. The desired Taylor series is then obtained:

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}} \qquad (|z-i|<\sqrt{2}).$$

9. The identity $\sinh(z+\pi i)=-\sinh z$ and the periodicity of $\sinh z$, with period $2\pi i$, tell us that

$$\sinh z=-\sinh(z+\pi i)=-\sinh(z-\pi i).$$

So, if we replace z by $z-\pi i$ in the known representation

$$\sinh z=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!} \qquad (|z|<\infty)$$

and then multiply through by -1, we find that

$$\sinh z=-\sum_{n=0}^{\infty}\frac{(z-\pi i)^{2n+1}}{(2n+1)!} \qquad (|z-\pi i|<\infty).$$

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(a) $$\frac{\sinh z}{z^2} = \frac{\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}}{z^2}$$

$$= \frac{z}{z^2} + \frac{\sum_{n=1}^{\infty} \frac{z^{2n+1}}{(2n+1)!}}{z^2}$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n+1)!} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!}, \quad (0 < |z| < \infty)$$

$n \to n+1$

(b)

$$z^3 \cosh\left(\frac{1}{z}\right) = z^3 \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^{2n} \frac{1}{(2n)!}$$

$$= z^3 + z^3 \left(\frac{1}{z}\right)^2 \frac{1}{2} + z^3 \sum_{n=2}^{\infty} \left(\frac{1}{z}\right)^{2n} \frac{1}{(2n)!}$$

$n \to n+1$

$$= z^3 + \frac{z}{2} + \sum_{n=1}^{\infty} \frac{1}{z^{2n-1}(2n+2)!} \quad \left(0 < |z| < \infty\right)$$