p. 205

HW 09 Solutions

1. We may use the expansion

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty)$$

to see that when $0 < |z| < \infty$,

$$z^2 \sin\left(\frac{1}{z^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}}.$$

2.

$$\frac{e^{z+1-1}}{(z+1)^2} = \frac{e^{-1}}{(z+1)^2} \sum_{n=0}^{\infty} \frac{(z+1)^n}{n!}$$

$$= e^{-1}\left[\frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=2}^{\infty} \frac{(z+1)^{n-2}}{n!}\right]$$

$$= e^{-1}\left[\frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!}\right] \qquad 0 < |z+1| < \infty$$

3. Suppose that $1 < |z| < \infty$ and recall the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1).$$

This enables us to write

$$\frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+\frac{1}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} \qquad (1 < |z| < \infty).$$

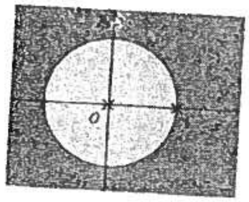
Replacing n by $n-1$ in this last series and then noting that

$$(-1)^{n-1} = (-1)^{n-1}(-1)^2 = (-1)^{n+1},$$

we arrive at the desired expansion:

$$\frac{1}{1+z} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n} \qquad (1 < |z| < \infty).$$

4. The singularities of the function $f(z) = \frac{1}{z^2(1-z)}$ are at the points $z = 0$ and $z = 1$. Hence there are Laurent series in powers of z for the domains $0 < |z| < 1$ and $1 < |z| < \infty$ (see the figure below).



To find the series when $0 < |z| < 1$, recall that $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ $(|z| < 1)$ and write

$$f(z) = \frac{1}{z^2} \cdot \frac{1}{1-z} = \frac{1}{z^2}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-2} = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=2}^{\infty} z^{n-2} = \sum_{n=0}^{\infty} z^n + \frac{1}{z} + \frac{1}{z^2}.$$

As for the domain $1 < |z| < \infty$, note that $|1/z| < 1$ and write

$$f(z) = -\frac{1}{z^3} \cdot \frac{1}{1-(1/z)} = -\frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=0}^{\infty}\frac{1}{z^{n+3}} = -\sum_{n=3}^{\infty}\frac{1}{z^n}.$$

6. $0 < |z-1| < 2$

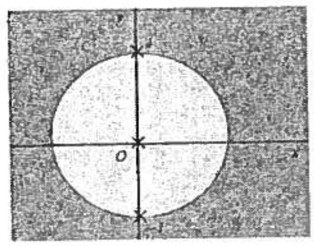
$$\frac{z}{(z-1)(z-3)} = \frac{z-3+3}{(z-1)(z-3)} = \frac{1}{z-1} + \frac{3}{(z-1)(-2+z-1)}$$

$$= \frac{1}{z-1} + \frac{3}{(z-1)(-2)\left(1-\frac{z-1}{2}\right)} = \frac{1}{z-1} + \frac{3}{(-2)(z-1)}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n}$$

$$= \frac{1}{z-1} + \frac{3}{(-2)(z-1)} + \frac{3}{(-2)}\sum_{n=1}^{\infty}\frac{(z-1)^{n-1}}{2^n}$$

$$= -\frac{1}{z-1} \quad 3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}$$

7. The function $f(z)=\dfrac{1}{z(1+z^2)}$ has isolated singularities at $z=0$ and $z=\pm i$, as indicated in the figure below. Hence there is a Laurent series representation for the domain $0<|z|<1$ and also one for the domain $1<|z|<\infty$, which is exterior to the circle $|z|=1$.



To find each of these Laurent series, we recall the Maclaurin series representation

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n \qquad (|z|<1).$$

For the domain $0<|z|<1$, we have

$$f(z)=\frac{1}{z}\cdot\frac{1}{1+z^2}=\frac{1}{z}\sum_{n=0}^{\infty}(-z^2)^n=\sum_{n=0}^{\infty}(-1)^n z^{2n-1}=\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n z^{2n-1}=\sum_{n=0}^{\infty}(-1)^{n+1} z^{2n+1}+\frac{1}{z}.$$

On the other hand, when $1<|z|<\infty$,

$$f(z)=\frac{1}{z^3}\cdot\frac{1}{1+\dfrac{1}{z^2}}=\frac{1}{z^3}\sum_{n=0}^{\infty}\left(-\frac{1}{z^2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+3}}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}.$$

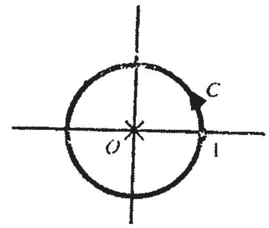
In this second expansion, we have used the fact that $(-1)^{n-1}=(-1)^{n-1}(-1)^2=(-1)^{n+1}$.

(Optional)

10. (d) Let z be any fixed complex number and C the unit circle $w=e^{i\phi}$ $(-\pi\le\phi\le\pi)$ in the w plane. The function

$$f(w)=\exp\left[\frac{z}{2}\left(w-\frac{1}{w}\right)\right]$$

has the one singularity $w=0$ in the w plane. That singularity is, of course, interior to C, as shown in the figure below.



w plane

Now the function $f(w)$ has a Laurent series representation in the domain $0<|w|<\infty$. According to expression (5), Sec. 55, then,

$$\exp\left[\frac{z}{2}\left(w-\frac{1}{w}\right)\right]=\sum_{n=-\infty}^{\infty}J_n(z)w^n \qquad (0<|w|<\infty),$$

where the coefficients $J_n(z)$ are

$$J_n(z)=\frac{1}{2\pi i}\int_C \frac{\exp\left[\frac{z}{2}\left(w-\frac{1}{w}\right)\right]}{w^{n+1}}dw \qquad (n=0,\pm1,\pm2,\ldots).$$

Using the parametric representation $w=e^{i\phi}$ $(-\pi\le\phi\le\pi)$ for C, let us rewrite this expression for $J_n(z)$ as follows:

$$J_n(z)=\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{\exp\left[\frac{z}{2}\left(e^{i\phi}-e^{-i\phi}\right)\right]}{e^{i(n+1)\phi}}ie^{i\phi}d\phi=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp[iz\sin\phi]e^{-in\phi}d\phi.$$

That is,

$$J_n(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp[-i(n\phi-z\sin\phi)]d\phi \qquad (n=0,\pm1,\pm2,\ldots).$$

(b) The last expression for $J_n(z)$ in part *(a)* can be written as

$$\begin{aligned}J_n(z)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}[\cos(n\phi-z\sin\phi)-i\sin(n\phi-z\sin\phi)]d\phi\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(n\phi-z\sin\phi)d\phi-\frac{i}{2\pi}\int_{-\pi}^{\pi}\sin(n\phi-z\sin\phi)d\phi\\&=\frac{1}{2\pi}2\int_{0}^{\pi}\cos(n\phi-z\sin\phi)d\phi-\frac{i}{2\pi}0 \qquad (n=0,\pm1,\pm2,\ldots).\end{aligned}$$

That is,

$$J_n(z)=\frac{1}{\pi}\int_0^{\pi}\cos(n\phi-z\sin\phi)d\phi \qquad (n=0,\pm1,\pm2,\ldots).$$