math 313 HW 10 Solutions

p. 219

1. Differentiating each side of the representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1),$$

we find that

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} \frac{d}{dz} z^n = \sum_{n=1}^{\infty} n z^{n-1} = \sum_{n=0}^{\infty} (n+1) z^n \qquad (|z| < 1).$$

Another differentiation gives

$$\frac{2}{(1-z)^3} = \frac{d}{dz}\sum_{n=0}^{\infty} (n+1) z^n = \sum_{n=0}^{\infty} (n+1)\frac{d}{dz} z^n = \sum_{n=1}^{\infty} n(n+1) z^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2) z^n \qquad (|z| < 1).$$

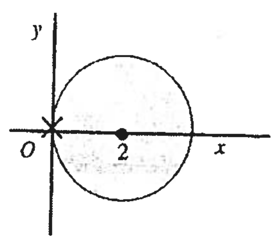
2. Replace z by $1/(1-z)$ on each side of the Maclaurin series representation (Exercise 1)

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n \qquad (|z| < 1),$$

as well as in its condition of validity. This yields the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^n (n-1)}{(z-1)^n} \qquad (1 < |z-1| < \infty).$$

3. Since the function $f(z) = 1/z$ has a singular point at $z = 0$, its Taylor series about $z_0 = 2$ is valid in the open disk $|z-2| < 2$, as indicated in the figure below.



To find that series, write

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2} \cdot \frac{1}{1+(z-2)/2}$$

to see that it can be obtained by replacing z by $-(z-2)/2$ in the known expansion

Specifically,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1).$$

or

$$\frac{1}{z} = \frac{1}{2}\sum_{n=0}^{\infty}\left[-\frac{(z-2)}{2}\right]^n \qquad (|z-2| < 2),$$

$$\frac{1}{z} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^n \qquad (|z-2| < 2).$$

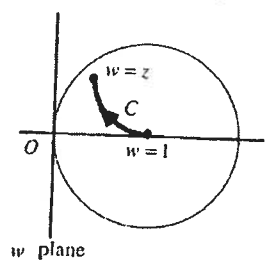
Differentiating this series term by term, we have

$$-\frac{1}{z^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{2^{n+1}}n(z-2)^{n-1} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{n+2}}(n+1)(z-2)^n \qquad (|z-2| < 2).$$

Thus

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \qquad (|z-2| < 2).$$

6. (optional) Let C be a contour lying in the open disk $|w-1| < 1$ in the w plane that extends from the point $w = 1$ to a point $w = z$, as shown in the figure below.



According to Theorem 1 in Sec. 65, we can integrate the Taylor series representation

$$\frac{1}{w} = \sum_{n=0}^{\infty}(-1)^n(w-1)^n \qquad (|w-1| < 1)$$

term by term along the contour C. Thus

$$\int_C \frac{dw}{w} = \int_C \sum_{n=0}^{\infty}(-1)^n(w-1)^n\,dw = \sum_{n=0}^{\infty}(-1)^n\int_C (w-1)^n\,dw.$$

But

$$\int_C \frac{dw}{w} = \int_1^z \frac{dw}{w} = [\operatorname{Log} w]_1^z = \operatorname{Log} z - \operatorname{Log} 1 = \operatorname{Log} z$$

and

$$\int_C (w-1)^n = \int_1^z (w-1)^n\,dw = \left[\frac{(w-1)^{n+1}}{n+1}\right]_1^z = \frac{(z-1)^{n+1}}{n+1}.$$

Hence

$$\operatorname{Log} z=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(z-1)^{n+1}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(z-1)^n \qquad (|z-1|<1);$$

and, since $(-1)^{n-1}=(-1)^{n-1}(-1)^2=(-1)^{n+1}$, this result becomes

$$\operatorname{Log} z=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \qquad (|z-1|<1).$$

12 (optional)

$$f_2(z)=\frac{1}{z^2}\quad(z\neq 0)$$

$$f_2(z)=\frac{1}{(z+1-1)^2}=\frac{1}{(1-(z+1))^2}=\frac{d}{dz}\frac{1}{1-(1+z)}$$

$$=\frac{d}{dz}\sum_{n=0}^{\infty}(1+z)^n=\sum_{n=1}^{\infty}n(1+z)^n$$

$$=\sum_{n=0}^{\infty}(n+1)(1+z)^{n+1}\quad(|z+1|<1)$$

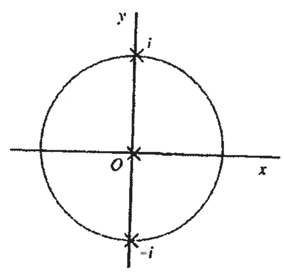
$\Rightarrow$ $f_1(z)=f_2(z)$ for $(|z+1|<1)$

$\Rightarrow$ from analytic continuation $f_1(z)=f_2(z)$ except $z=0$.

①

p.225

1. The singularities of the function $f(z)=\dfrac{e^z}{z(z^2+1)}$ are at $z=0,\pm i$. The problem here is to find the Laurent series for f that is valid in the punctured disk $0<|z|<1$, shown below.



We begin by recalling the Maclaurin series representations

$$e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots \qquad (|z|<\infty)$$

and

$$\frac{1}{1-z}=1+z+z^2+z^3+\cdots \qquad (|z|<1),$$

which enable us to write

$$e^z=1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\cdots \qquad (|z|<\infty)$$

and

$$\frac{1}{z^2+1}=1-z^2+z^4-z^6+\cdots \qquad (|z|<1).$$

Multiplying these last two series term by term, we have the Maclaurin series representation

$$\begin{aligned}\frac{e^z}{z^2+1}=1+z+\frac{1}{2}z^2&+\frac{1}{6}z^3+\cdots\\ -z^2&-z^3-\cdots\\ &\qquad z^4+\cdots\\ &\qquad\quad\vdots\\ =1+z-\frac{1}{2}z^2&-\frac{5}{6}z^3+\cdots,\end{aligned}$$

which is valid when $|z|<1$. The desired Laurent series is then obtained by multiplying each side of the above representation by $\dfrac{1}{z}$:

$$\frac{e^z}{z(z^2+1)}=\frac{1}{z}+1-\frac{1}{2}z-\frac{5}{6}z^2+\cdots \qquad (0<|z|<1).$$

(2)

(2)

$$\csc z = \frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}$$

$$= \frac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots - \left(-\frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)}{z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!}\right)}$$

$$= \frac{1}{z} + \frac{\frac{z^2}{3!}}{z} \frac{\left[1 - \frac{z^2 3!}{5!} + \cdots\right]}{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)}$$

$$= \frac{1}{z} + \frac{z}{3!} \frac{\left[\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right) + \frac{z^2}{3!} - \frac{z^4}{5!} - \frac{z^2 3!}{5!} + \cdots\right]}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots}$$

$$= \frac{1}{z} + \frac{z}{3!} + \frac{z \cdot z^2}{3!}\left(\frac{1}{3!} - \frac{3!}{5!}\right) + \cdots$$

$$= \frac{1}{z} + \frac{z}{3!} + z^3\left[\frac{1}{(3!)^2} - \frac{1}{5!}\right] + \cdots$$

5 (optional)

(a) $$\frac{1}{1 + z^2/3! + z^4/5! + \cdots} = d_0 + d_1 z + d_2 z^2 + \cdots$$

$$\Rightarrow \quad 1 = \left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)\left(d_0 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + \cdots\right)$$

$$= d_0 + d_1 z + z^2\left(d_2 + \frac{1}{3!} d_0\right) + \left(d_3 + \frac{d_1}{3!}\right) z^3$$

$$+ \left(d_4 + \frac{d_2}{3!} + \frac{d_0}{5!}\right) z^4 + \cdots = 0$$

$$|z| < \pi$$

$$\Rightarrow \begin{array}{c|l} z^0 & d_0 = 1 \\ z^1 & d_1 = 0 \\ z^2 & d_2 = -\frac{1}{3!} d_0 = -\frac{1}{3!} = -\frac{1}{6} \\ z^3 & d_3 + 0 = 0 \Rightarrow d_3 = 0 \\ z^4 & d_4 + \frac{d_2}{3!} + \frac{d_0}{5!} = d_4 - \frac{1}{(3!)^2} + \frac{1}{5!} = 0 \end{array}$$

$$\Rightarrow \quad d_4 = \frac{1}{(3!)^2} - \frac{1}{5!} = \frac{7}{360}$$

math 313 HW 10 Solutions

1

p 239

1. (a) Let us write

$$\frac{1}{z+z^2}=\frac{1}{z}\cdot\frac{1}{1+z}=\frac{1}{z}(1-z+z^2-z^3+\cdots)=\frac{1}{z}-1+z-z^2+\cdots \qquad (0<|z|<1).$$

The residue at $z=0$, which is the coefficient of $\frac{1}{z}$, is clearly 1.

(b) We may use the expansion

$$\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots \qquad (|z|<\infty)$$

to write

$$z\cos\left(\frac{1}{z}\right)=z\left(1-\frac{1}{2!}\cdot\frac{1}{z^2}+\frac{1}{4!}\cdot\frac{1}{z^4}-\frac{1}{6!}\cdot\frac{1}{z^6}+\cdots\right)=z-\frac{1}{2!}\cdot\frac{1}{z}+\frac{1}{4!}\cdot\frac{1}{z^3}-\frac{1}{6!}\cdot\frac{1}{z^5}+\cdots$$

$$(0<|z|<\infty).$$

The residue at $z=0$, or coefficient of $\frac{1}{z}$, is now seen to be $-\frac{1}{2}$.

(c) Observe that

$$\frac{z-\sin z}{z}=\frac{1}{z}(z-\sin z)=\frac{1}{z}\left[z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)\right]=\frac{z^2}{3!}-\frac{z^4}{5!}+\cdots \qquad (0<|z|<\infty).$$

Since the coefficient of $\frac{1}{z}$ in this Laurent series is 0, the residue at $z=0$ is 0.

math 313 Hw 10 Solutions. ②

p. 239

2. In each part, C denotes the positively oriented circle $|z| = 3$.

(a) To evaluate $\int_C \frac{\exp(-z)}{z^2}\,dz$, we need the residue of the integrand at $z = 0$. From the Laurent series

$$\frac{\exp(-z)}{z^2} = \frac{1}{z^2}\left(1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right) = \frac{1}{z^2} - \frac{1}{1!}\cdot\frac{1}{z} + \frac{1}{2!} - \frac{z}{3!} + \cdots \qquad (0 < |z| < \infty),$$

we see that the required residue is -1. Thus

$$\int_C \frac{\exp(-z)}{z^2}\,dz = 2\pi i(-1) = -2\pi i.$$

(b) $|z| = 3$

$$f(z) = \frac{e^{-(z-1)-1}}{(z-1)^2} = \frac{1 - (z-1) + \frac{(z-1)^2}{2!} + \cdots}{(z-1)^2}\, e^{-1} \qquad 0 < |z-1| < \infty$$

$$\Rightarrow \operatorname*{Res}_{z=1} \frac{e^{-z}}{(z-1)^2} = -e^{-1}$$

$$\Rightarrow \int_C f(z)\,dz = -2\pi i\, e^{-1}$$

(c) Likewise, to evaluate the integral $\int_C z^2 \exp\left(\frac{1}{z}\right)dz$, we must find the residue of the integrand at $z=0$. The Laurent series

$$z^2 \exp\left(\frac{1}{z}\right) = z^2\left(1+\frac{1}{1!}\cdot\frac{1}{z}+\frac{1}{2!}\cdot\frac{1}{z^2}+\frac{1}{3!}\cdot\frac{1}{z^3}+\frac{1}{4!}\cdot\frac{1}{z^4}+\cdots\right)$$

$$= z^2+\frac{z}{1!}+\frac{1}{2!}+\frac{1}{3!}\cdot\frac{1}{z}+\frac{1}{4!}\cdot\frac{1}{z^2}+\cdots,$$

which is valid for $0<|z|<\infty$, tells us that the needed residue is $\frac{1}{6}$. Hence

$$\int_C z^2 \exp\left(\frac{1}{z}\right)dz = 2\pi i\left(\frac{1}{6}\right)=\frac{\pi i}{3}.$$

3. In each part of this problem, C is the positively oriented circle $|z|=2$.

(a) If $f(z)=\frac{z^5}{1-z^3}$, then

$$\frac{1}{z^2}f\left(\frac{1}{z}\right)=\frac{1}{z^7-z^4}=-\frac{1}{z^4}\cdot\frac{1}{1-z^3}=-\frac{1}{z^4}\left(1+z^3+z^6+\cdots\right)=-\frac{1}{z^4}-\frac{1}{z}-z^2-\cdots$$

when $0<|z|<1$. This tells us that

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right)=2\pi i(-1)=-2\pi i.$$

4

4. Let C denote the circle $|z| = 1$, taken counterclockwise.

(a) The Maclaurin series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $(|z| < \infty)$ enables us to write

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = \int_C e^z e^{1/z} dz = \int_C e^{1/z} \sum_{n=0}^{\infty} \frac{z^n}{n!} dz = \sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n \exp\left(\frac{1}{z}\right) dz.$$

(b) Referring to the Maclaurin series for e^z once again, let us write

$$z^n \exp\left(\frac{1}{z}\right) = z^n \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{z^k} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{n-k} \qquad (n = 0, 1, 2, \ldots).$$

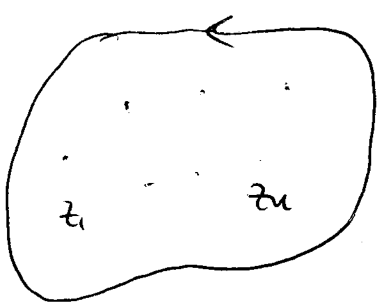
Now the $\frac{1}{z}$ in this series occurs when $n - k = -1$, or $k = n + 1$. So, by the residue theorem,

$$\int_C z^n \exp\left(\frac{1}{z}\right) dz = 2\pi i \frac{1}{(n+1)!} \qquad (n = 0, 1, 2, \ldots).$$

The final result in part *(a)* thus reduces to

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}.$$

5 (optional)



$$I = \int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) \qquad (1)$$

But at the same time

$$I = \int_C f(z)\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z) \qquad (2)$$

from sec 71

$\Rightarrow (1) = (2) \Rightarrow$ ☒

5

~~p. 242~~ p. 243

1. (a) From the expansion

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (|z| < \infty),$$

we see that

$$z\exp\left(\frac{1}{z}\right) = z + 1 + \frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{3!}\cdot\frac{1}{z^2} + \cdots \qquad (0 < |z| < \infty).$$

The principal part of $z\exp\left(\frac{1}{z}\right)$ at the isolated singular point $z = 0$ is, then,

$$\frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{3!}\cdot\frac{1}{z^2} + \cdots;$$

and $z = 0$ is an essential singular point of that function.

(b) The isolated singular point of $\frac{z^2}{1+z}$ is at $z = -1$. Since the principal part at $z = -1$ involves powers of $z + 1$, we begin by observing that

$$z^2 = (z+1)^2 - 2z - 1 = (z+1)^2 - 2(z+1) + 1.$$

This enables us to write

$$\frac{z^2}{1+z} = \frac{(z+1)^2 - 2(z+1) + 1}{z+1} = (z+1) - 2 + \frac{1}{z+1}.$$

Since the principal part is $\frac{1}{z+1}$, the point $z = -1$ is a (simple) pole.

(c) The point $z = 0$ is the isolated singular point of $\frac{\sin z}{z}$, and we can write

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \qquad (0 < |z| < \infty).$$

The principal part here is evidently 0, and so $z = 0$ is a removable singular point of the function $\frac{\sin z}{z}$.

(d) The isolated singular point of $\dfrac{\cos z}{z}$ is $z=0$. Since

$$\frac{\cos z}{z}=\frac{1}{z}\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)=\frac{1}{z}-\frac{z}{2!}+\frac{z^3}{4!}-\cdots \qquad (0<|z|<\infty),$$

the principal part is $\dfrac{1}{z}$. This means that $z=0$ is a (simple) pole of $\dfrac{\cos z}{z}$.

(e) Upon writing $\dfrac{1}{(2-z)^3}=\dfrac{-1}{(z-2)^3}$, we find that the principal part of $\dfrac{1}{(2-z)^3}$ at its isolated singular point $z=2$ is simply the function itself. That point is evidently a pole (of order 3).

2. *(a)* The singular point is $z=0$. Since

$$\frac{1-\cosh z}{z^3}=\frac{1}{z^3}\left[1-\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\cdots\right)\right]=-\frac{1}{2!}\cdot\frac{1}{z}-\frac{z}{4!}-\frac{z^3}{6!}-\cdots$$

when $0<|z|<\infty$, we have $m=1$ and $B=-\dfrac{1}{2!}=-\dfrac{1}{2}$.

(b) Here the singular point is also $z=0$. Since

$$\frac{1-\exp(2z)}{z^4}=\frac{1}{z^4}\left[1-\left(1+\frac{2z}{1!}+\frac{2^2z^2}{2!}+\frac{2^3z^3}{3!}+\frac{2^4z^4}{4!}+\frac{2^5z^5}{5!}+\cdots\right)\right]$$
$$=-\frac{2}{1!}\cdot\frac{1}{z^3}-\frac{2^2}{2!}\cdot\frac{1}{z^2}-\frac{2^3}{3!}\cdot\frac{1}{z}-\frac{2^4}{4!}-\frac{2^5}{5!}z-\cdots$$

when $0<|z|<\infty$, we have $m=3$ and $B=-\dfrac{2^3}{3!}=-\dfrac{4}{3}$.

(c) The singular point of $\dfrac{\exp(2z)}{(z-1)^2}$ is $z=1$. The Taylor series

$$\exp(2z)=e^{2(z-1)}e^2=e^2\left[1+\frac{2(z-1)}{1!}+\frac{2^2(z-1)^2}{2!}+\frac{2^3(z-1)^3}{3!}+\cdots\right] \qquad (|z|<\infty)$$

enables us to write the Laurent series

$$\frac{\exp(2z)}{(z-1)^2}=e^2\left[\frac{1}{(z-1)^2}+\frac{2}{1!}\cdot\frac{1}{z-1}+\frac{2^2}{2!}+\frac{2^2}{3!}(z-1)+\cdots\right] \qquad (0<|z-1|<\infty).$$

Thus $m=2$ and $B=e^2\dfrac{2}{1!}=2e^2$.

3. Since f is analytic at z_0, it has a Taylor series representation

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots \qquad (|z - z_0| < R_0).$$

Let g be defined by means of the equation

$$g(z) = \frac{f(z)}{z - z_0}.$$

(a) Suppose that $f(z_0) \neq 0$. Then

$$g(z) = \frac{1}{z - z_0}\left[f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots\right]$$

$$= \frac{f(z_0)}{z - z_0} + \frac{f'(z_0)}{1!} + \frac{f''(z_0)}{2!}(z - z_0) + \cdots \qquad (0 < |z - z_0| < R_0).$$

This shows that g has a simple pole at z_0, with residue $f(z_0)$.

(b) Suppose, on the other hand, that $f(z_0) = 0$. Then

$$g(z) = \frac{1}{z - z_0}\left[\frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots\right]$$

$$= \frac{f'(z_0)}{1!} + \frac{f''(z_0)}{2!}(z - z_0) + \cdots \qquad (0 < |z - z_0| < R_0).$$

Since the principal part of g at z_0 is just 0, the point $z = 0$ is a removable singular point of g.

5. Write the function

$$f(z) = \frac{8a^3z^2}{(z^2 + a^2)^3} \qquad (a > 0)$$

as

$$f(z) = \frac{\phi(z)}{(z - ai)^3} \quad \text{where} \quad \phi(z) = \frac{8a^3z^2}{(z + ai)^3}.$$

Since the only singularity of $\phi(z)$ is at $z = -ai$, $\phi(z)$ has a Taylor series representation

$$\phi(z) = \phi(ai) + \frac{\phi'(ai)}{1!}(z - ai) + \frac{\phi''(ai)}{2!}(z - ai)^2 + \cdots \qquad (|z - ai| < 2a)$$

about $z = ai$. Thus

$$f(z) = \frac{1}{(z-ai)^3}\left[\phi(ai) + \frac{\phi'(ai)}{1!}(z-ai) + \frac{\phi''(ai)}{2!}(z-ai)^2 + \cdots\right] \qquad (0 < |z-ai| < 2a).$$

Now straightforward differentiation reveals that

$$\phi'(z) = \frac{16a^4 iz - 8a^3 z^2}{(z+ai)^4} \quad \text{and} \quad \phi''(z) = \frac{16a^3(z^2 - 4aiz - a^2)}{(z+ai)^5}.$$

Consequently,

$$\phi(ai) = -a^2 i, \quad \phi'(ai) = -\frac{a}{2}, \quad \text{and} \quad \phi''(ai) = -i.$$

This enables us to write

$$f(z) = \frac{1}{(z-ai)^3}\left[-a^2 i - \frac{a}{2}(z-ai) - \frac{i}{2}(z-ai)^2 + \cdots\right] \qquad (0 < |z-ai| < 2a).$$

The principal part of f at the point $z = ai$ is, then,

$$-\frac{i/2}{z-ai} - \frac{a/2}{(z-ai)^2} - \frac{a^2 i}{(z-ai)^3}.$$