Math 313 HW 11 Solutions

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1. (a) The function $f(z)=\frac{z^2+2}{z-1}$ has an isolated singular point at $z=1$. Writing $f(z)=\frac{\phi(z)}{z-1}$, where $\phi(z)=z^2+2$, and observing that $\phi(z)$ is analytic and nonzero at $z=1$, we see that $z=1$ is a pole of order $m=1$ and that the residue there is $B=\phi(1)=3$.

(b) If we write

$$f(z)=\left(\frac{z}{2z+1}\right)^3=\frac{\phi(z)}{\left[z-\left(-\frac{1}{2}\right)\right]^3}, \quad \text{where} \quad \phi(z)=\frac{z^3}{8},$$

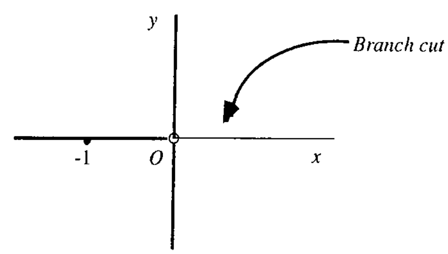
we see that $z=-\frac{1}{2}$ is a singular point of f. Since $\phi(z)$ is analytic and nonzero at that point, f has a pole of order $m=3$ there. The residue is

$$B=\frac{\phi''(-1/2)}{2!}=-\frac{3}{16}.$$

(2.) *(a)* Write the function $f(z) = \frac{z^{1/4}}{z+1}$ ($|z| > 0, 0 < \arg z < 2\pi$) as

$$f(z) = \frac{\phi(z)}{z+1}, \quad \text{where} \quad \phi(z) = z^{1/4} = e^{\frac{1}{4}\log z} \; (|z| > 0, 0 < \arg z < 2\pi).$$

The function $\phi(z)$ is analytic throughout its domain of definition, indicated in the figure below.



Also,

$$\phi(-1) = (-1)^{1/4} = e^{\frac{1}{4}\log(-1)} = e^{\frac{1}{4}(\ln 1 + i\pi)} = e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1+i}{\sqrt{2}} \neq 0.$$

This shows that the function f has a pole of order $m = 1$ at $z = -1$, the residue there being

$$B = \phi(-1) = \frac{1+i}{\sqrt{2}}.$$

(b) Write the function $f(z) = \frac{\text{Log}\, z}{(z^2+1)^2}$ as

$$f(z) = \frac{\phi(z)}{(z-i)^2} \quad \text{where} \quad \phi(z) = \frac{\text{Log}\, z}{(z+i)^2}.$$

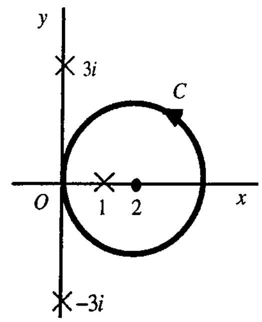
From this, it is clear that $f(z)$ has a pole of order $m = 2$ at $z = i$. Straightforward differentiation then reveals that

$$\operatorname*{Res}_{z=i} \frac{\text{Log}\, z}{(z^2+1)^2} = \phi'(i) = \frac{\pi + 2i}{8}.$$

3. (a) We wish to evaluate the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz,$$

where C is the circle $|z-2|=2$, taken in the counterclockwise direction. That circle and the singularities $z=1, \pm 3i$ of the integrand are shown in the figure just below.



Observe that the point $z=1$, which is the only singularity inside C, is a simple pole of the integrand and that

$$\underset{z=1}{\text{Res}}\frac{3z^3+2}{(z-1)(z^2+9)}=\left.\frac{3z^3+2}{z^2+9}\right]_{z=1}=\frac{1}{2}.$$

According to the residue theorem, then,

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz=2\pi i\left(\frac{1}{2}\right)=\pi i.$$

5. Let us evaluate the integral $\int_C \frac{\cosh \pi z\, dz}{z(z^2+1)}$, where C is the positively oriented circle $|z|=2$. All three isolated singularities $z=0, \pm i$ of the integrand are interior to C. The desired residues are

$$\underset{z=0}{\text{Res}}\frac{\cosh \pi z}{z(z^2+1)}=\left.\frac{\cosh \pi z}{z^2+1}\right]_{z=0}=1,$$

$$\underset{z=i}{\text{Res}}\frac{\cosh \pi z}{z(z^2+1)}=\left.\frac{\cosh \pi z}{z(z+i)}\right]_{z=i}=\frac{1}{2},$$

and

$$\underset{z=-i}{\text{Res}}\frac{\cosh \pi z}{z(z^2+1)}=\left.\frac{\cosh \pi z}{z(z-i)}\right]_{z=-i}=\frac{1}{2}.$$

Consequently,

$$\int_C \frac{\cosh \pi z\, dz}{z(z^2+1)}=2\pi i\left(1+\frac{1}{2}+\frac{1}{2}\right)=4\pi i.$$

(A)

6. In each part of this problem, C denotes the positively oriented circle $|z| = 3$.

(a) It is straightforward to show that

$$\text{if } f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}, \quad \text{then} \quad \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{(3+2z)^2}{z(1-z)(2+5z)}.$$

This function $\frac{1}{z^2} f\left(\frac{1}{z}\right)$ has a simple pole at $z = 0$, and

$$\int_C \frac{(3z+2)^2}{z(z-1)(2z+5)} dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 2\pi i\left(\frac{9}{2}\right) = 9\pi i.$$

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(1) (a) we use that

$$\csc z = \frac{1}{\sin z} = \frac{p(z)}{q(z)}, \text{ where } p(z)=1, \; q(z)=\sin z$$

~~Because~~ Since $p(0)=1\neq 0$ while $q(0)=\sin 0 = 0$ but

$$q'(0)=\cos 0 = 1 \neq 0$$

$\Rightarrow z=0$ is the simple pole of $\frac{1}{\sin z}$

$$\Rightarrow \operatorname*{Res}_{z=0} \frac{1}{\sin z} = \frac{p(0)}{q'(0)} = \frac{1}{1} = 1$$

(b) From Ex. 2, Sec 67 we use that

$$\csc z = \frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^3 + \cdots, \quad 0<|z|<\pi$$

[to show that: $\frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{3!} + \cdots} = \frac{1}{z(1-\frac{z^2}{3!}+\cdots)} = \frac{1}{z}(1+\frac{z^2}{3!}+\cdots)$]

Since the coefficient of $\frac{1}{z}$ here is 1

$\Rightarrow z=0$ is a simple pole of $\csc z$

with $\operatorname*{Res}_{z=0} \csc z = 1$

2. (a) Write

$$\frac{z-\sinh z}{z^2\sinh z}=\frac{p(z)}{q(z)}, \quad \text{where} \quad p(z)=z-\sinh z \text{ and } q(z)=z^2\sinh z.$$

Since

$$p(\pi i)=\pi i\neq 0, \quad q(\pi i)=0, \quad \text{and} \quad q'(\pi i)=\pi^2\neq 0,$$

it follows that

$$\operatorname*{Res}_{z=\pi i}\frac{z-\sinh z}{z^2\sinh z}=\frac{p(\pi i)}{q'(\pi i)}=\frac{\pi i}{\pi^2}=\frac{i}{\pi}.$$

3. (a) Write

$$f(z)=\frac{p(z)}{q(z)}, \quad \text{where} \quad p(z)=z \text{ and } q(z)=\cos z.$$

Observe that

$$q\left(\frac{\pi}{2}+n\pi\right)=0 \qquad (n=0,\pm1,\pm2,\ldots).$$

Also, for the stated values of n,

$$p\left(\frac{\pi}{2}+n\pi\right)=\frac{\pi}{2}+n\pi\neq 0 \quad \text{and} \quad q'\left(\frac{\pi}{2}+n\pi\right)=-\sin\left(\frac{\pi}{2}+n\pi\right)=(-1)^{n+1}\neq 0.$$

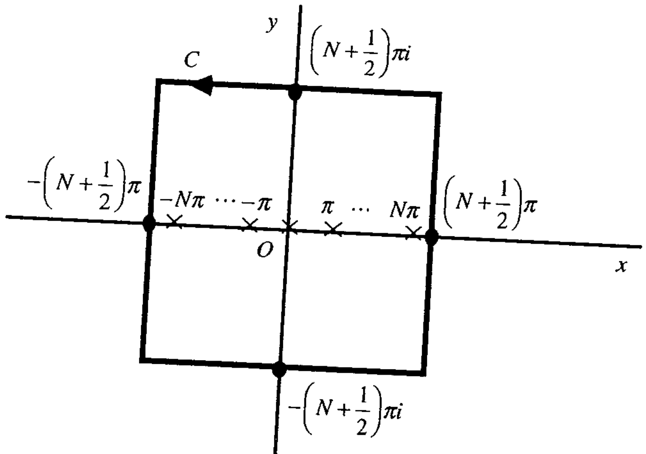
So the function $f(z)=\dfrac{z}{\cos z}$ has poles of order $m=1$ at each of the points

$$z_n=\frac{\pi}{2}+n\pi \qquad (n=0,\pm1,\pm2,\ldots).$$

The corresponding residues are

$$B=\frac{p(z_n)}{q'(z_n)}=(-1)^{n+1}z_n.$$

5. The simple closed contour C_N is as shown in the figure below.



Within C_N, the function $\dfrac{1}{z^2 \sin z}$ has isolated singularities at

$$z = 0 \quad \text{and} \quad z = \pm n\pi \ (n = 1, 2, \ldots, N).$$

To find the residue at $z = 0$, we recall the Laurent series for $\csc z$ that was found in Exercise 2, Sec. 67, and write

$$\frac{1}{z^2 \sin z} = \frac{1}{z^2}\csc z = \frac{1}{z^2}\left\{\frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^3 + \cdots\right\}$$

$$= \frac{1}{z^3} + \frac{1}{6}\cdot\frac{1}{z} + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z + \cdots \qquad (0 < |z| < \pi).$$

This tells us that $\dfrac{1}{z^2 \sin z}$ has a pole of order 3 at $z = 0$ and that

$$\operatorname*{Res}_{z=0} \frac{1}{z^2 \sin z} = \frac{1}{6}.$$

As for the points $z = \pm n\pi$ $(n = 1, 2, \ldots, N)$, write

$$\frac{1}{z^2 \sin z} = \frac{p(z)}{q(z)}, \quad \text{where} \quad p(z) = 1 \text{ and } q(z) = z^2 \sin z.$$

Since

$$p(\pm n\pi) = 1 \neq 0, \quad q(\pm n\pi) = 0, \quad \text{and} \quad q'(\pm n\pi) = n^2\pi^2 \cos n\pi = (-1)^n n^2\pi^2 \neq 0,$$

it follows that

$$\operatorname*{Res}_{z=\pm n\pi} \frac{1}{z^2 \sin z} = \frac{1}{(-1)^n n^2\pi^2} \cdot \frac{(-1)^n}{(-1)^n} = \frac{(-1)^n}{n^2\pi^2}.$$

So, by the residue theorem,

$$\int_{C_N} \frac{dz}{z^2 \sin z} dz = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2} \right].$$

Rewriting this equation in the form

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} - \frac{\pi}{4i} \int_{C_N} \frac{dz}{z^2 \sin z}$$

and recalling from Exercise 8, Sec. 43, that the value of the integral here tends to zero as N tends to infinity, we arrive at the desired summation formula:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

6. The path C here is the positively oriented boundary of the rectangle with vertices at the points ± 2 and $\pm 2 + i$. The problem is to evaluate the integral

$$\int_C \frac{dz}{(z^2 - 1)^2 + 3}.$$

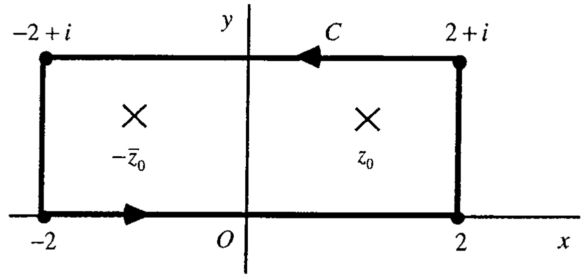
The isolated singularities of the integrand are the zeros of the polynomial

$$q(z) = (z^2 - 1)^2 + 3.$$

Setting this polynomial equal to zero and solving for z^2, we find that any zero z of $q(z)$ has the property $z^2 = 1 \pm \sqrt{3}i$. It is straightforward to find the two square roots of $1 + \sqrt{3}i$ and also the two square roots of $1 - \sqrt{3}i$. These are the four zeros of $q(z)$. Only two of those zeros,

$$z_0 = \sqrt{2}e^{i\pi/6} = \frac{\sqrt{3} + i}{\sqrt{2}} \quad \text{and} \quad -\bar{z}_0 = -\sqrt{2}e^{-i\pi/6} = \frac{-\sqrt{3} + i}{\sqrt{2}},$$

lie inside C. They are shown in the figure below.



To find the residues at z_0 and $-\bar{z}_0$, we write the integrand of the integral to be evaluated as

$$\frac{1}{(z^2-1)^2+3}=\frac{p(z)}{q(z)}, \quad \text{where} \quad p(z)=1 \text{ and } q(z)=(z^2-1)^2+3.$$

This polynomial $q(z)$ is, of course, the same $q(z)$ as above; hence $q(z_0)=0$. Note, too, that p and q are analytic at z_0 and that $p(z_0)\neq 0$. Finally, it is straightforward to show that $q'(z)=4z\left(z^2-1\right)$ and hence that

$$q'(z_0)=4z_0\left(z_0^2-1\right)=-2\sqrt{6}+6\sqrt{2}i\neq 0.$$

We may conclude, then, that z_0 is a simple pole of the integrand, with residue

$$\frac{p(z_0)}{q'(z_0)}=\frac{1}{-2\sqrt{6}+6\sqrt{2}i}.$$

Similar results are to be found at the singular point $-\bar{z}_0$. To be specific, it is easy to see that

$$q'(-\bar{z}_0)=-q'(\bar{z}_0)=-\overline{q'(z_0)}=2\sqrt{6}+6\sqrt{2}i\neq 0,$$

the residue of the integrand at $-\bar{z}_0$ being

$$\frac{p(-\bar{z}_0)}{q'(-\bar{z}_0)}=\frac{1}{2\sqrt{6}+6\sqrt{2}i}.$$

Finally, by the residue theorem,

$$\int_C \frac{dz}{(z^2-1)^2+3}=2\pi i\left(\frac{1}{-2\sqrt{6}+6\sqrt{2}i}+\frac{1}{2\sqrt{6}+6\sqrt{2}i}\right)=\frac{\pi}{2\sqrt{2}}.$$

(optional)

7. We are given that $f(z)=1/[q(z)]^2$, where q is analytic at z_0, $q(z_0)=0$, and $q'(z_0)\neq 0$. These conditions on q tell us that q has a zero of order $m=1$ at z_0. Hence $q(z)=(z-z_0)g(z)$, where g is a function that is analytic and nonzero at z_0; and this enables us to write

$$f(z)=\frac{\phi(z)}{(z-z_0)^2}, \quad \text{where} \quad \phi(z)=\frac{1}{[g(z)]^2}.$$

So f has a pole of order 2 at z_0, and

$$\operatorname*{Res}_{z=z_0} f(z)=\phi'(z_0)=-\frac{2g'(z_0)}{[g(z_0)]^3}.$$

But, since $q(z)=(z-z_0)g(z)$, we know that

$$q'(z)=(z-z_0)g'(z)+g(z) \quad \text{and} \quad q''(z)=(z-z_0)g''(z)+2g'(z).$$

Then, by setting $z=z_0$ in these last two equations, we find that

$$q'(z_0)=g(z_0) \quad \text{and} \quad q''(z_0)=2g'(z_0).$$

Consequently, our expression for the residue of f at z_0 can be put in the desired form:

$$\operatorname*{Res}_{z=0} f(z)=-\frac{q''(z_0)}{[q'(z_0)]^3}.$$

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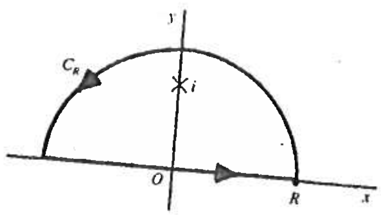
(1.) To evaluate the integral $\int_0^\infty \frac{dx}{x^2+1}$, we integrate the function $f(z) = \frac{1}{z^2+1}$ around the simple closed contour shown below, where $R > 1$.



We see that

$$\int_{-R}^{R} \frac{dx}{x^2+1} + \int_{C_R} \frac{dz}{z^2+1} = 2\pi i B,$$

where

$$B = \operatorname*{Res}_{z=i} \frac{1}{z^2+1} = \operatorname*{Res}_{z=i} \frac{1}{(z-i)(z+i)} = \left.\frac{1}{z+i}\right]_{z=i} = \frac{1}{2i}.$$

Thus

$$\int_{-R}^{R} \frac{dx}{x^2+1} = \pi - \int_{C_R} \frac{dz}{z^2+1}.$$

Now if z is a point on C_R,

$$|z^2+1| \geq ||z|^2 - 1| = R^2 - 1;$$

and so

$$\left| \int_{C_R} \frac{dz}{z^2+1} \right| \leq \frac{\pi R}{R^2-1} = \frac{\frac{\pi}{R}}{1 - \frac{1}{R^2}} \to 0 \quad \text{as} \quad R \to \infty.$$

Finally, then

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi, \quad \text{or} \quad \int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}.$$

(2)

(2.) The integral $\int_0^\infty \frac{dx}{(x^2+1)^2}$ can be evaluated using the function $f(z)=\frac{1}{(z^2+1)^2}$ and the same simple closed contour as in Exercise 1. Here

$$\int_{-R}^{R} \frac{dx}{(x^2+1)^2}+\int_{C_R} \frac{dz}{(z^2+1)^2}=2\pi i B,$$

where $B=\operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^2}$. Since

$$\frac{1}{(z^2+1)^2}=\frac{\phi(z)}{(z-i)^2}, \quad \text{where} \quad \phi(z)=\frac{1}{(z+i)^2},$$

we readily find that $B=\phi'(i)=\frac{1}{4i}$, and so

$$\int_{-R}^{R} \frac{dx}{(x^2+1)^2}=\frac{\pi}{2}-\int_{C_R} \frac{dz}{(z^2+1)^2}.$$

If z is a point on C_R, we know from Exercise 1 that

$$|z^2+1|\geq R^2-1;$$

thus

$$\left|\int_{C_R} \frac{dz}{(z^2+1)^2}\right| \leq \frac{\pi R}{(R^2-1)^2}=\frac{\frac{\pi}{R^3}}{\left(1-\frac{1}{R^2}\right)^2} \to 0 \quad \text{as} \quad R\to\infty.$$

The desired result is, then,

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}=\frac{\pi}{2}, \quad \text{or} \quad \int_0^\infty \frac{dx}{(x^2+1)^2}=\frac{\pi}{4}.$$

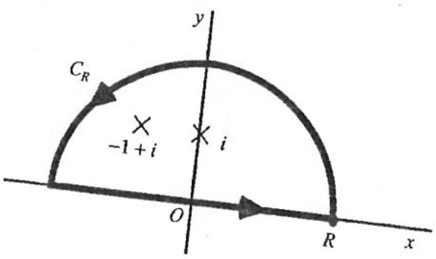
(7.) In order to show that

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{x\,dx}{(x^2+1)(x^2+2x+2)}=-\frac{\pi}{5},$$

we introduce the function

$$f(z)=\frac{z}{(z^2+1)(z^2+2z+2)}$$

and the simple closed contour shown below.



Observe that the singularities of $f(z)$ are at i, $z_0=-1+i$ and their conjugates $-i$, $\bar{z}_0=-1-i$ in the lower half plane. Also, if $R>\sqrt{2}$, we see that

$$\int_{-R}^{R} f(x)\,dx+\int_{C_R} f(z)\,dz=2\pi i(B_0+B_1),$$

where

$$B_0=\operatorname*{Res}_{z=z_0} f(z)=\left[\frac{z}{(z^2+1)(z-\bar{z}_0)}\right]_{z=z_0}=-\frac{1}{10}+\frac{3}{10}i$$

and

$$B_1=\operatorname*{Res}_{z=i} f(z)=\left[\frac{z}{(z+i)(z^2+2z+2)}\right]_{z=i}=\frac{1}{10}-\frac{1}{5}i.$$

Evidently, then,

$$\int_{-R}^{R}\frac{x\,dx}{(x^2+1)(x^2+2x+2)}=-\frac{\pi}{5}-\int_{C_R}\frac{z\,dz}{(z^2+1)(z^2+2z+2)}.$$

Since

$$\left|\int_{C_R}\frac{z\,dz}{(z^2+1)(z^2+2z+2)}\right|=\left|\int_{C_R}\frac{z\,dz}{(z^2+1)(z-z_0)(z-\bar{z}_0)}\right|\le\frac{\pi R^2}{(R^2-1)(R-\sqrt{2})^2}\to 0$$

as $R\to\infty$, this means that

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x\,dx}{(x^2+1)(x^2+2x+2)}=-\frac{\pi}{5}.$$

This is the desired result.

9. (optional) Let m and n be integers, where $0 \le m < n$. The problem here is to derive the integration formula

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}dx = \frac{\pi}{2n}\csc\left(\frac{2m+1}{2n}\pi\right).$$

(a) The zeros of the polynomial $z^{2n}+1$ occur when $z^{2n}=-1$. Since

$$(-1)^{1/(2n)} = \exp\left[i\frac{(2k+1)\pi}{2n}\right] \qquad (k=0,1,2,\ldots,2n-1),$$

it is clear that the zeros of $z^{2n}+1$ in the upper half plane are

$$c_k = \exp\left[i\frac{(2k+1)\pi}{2n}\right] \qquad (k=0,1,2,\ldots,n-1)$$

and that there are none on the real axis.

(b) With the aid of Theorem 2 in Sec. 76, we find that

$$\operatorname*{Res}_{z=c_k}\frac{z^{2m}}{z^{2n}+1} = \frac{c_k^{2m}}{2nc_k^{2n-1}} = \frac{1}{2n}c_k^{2(m-n)+1} \qquad (k=0,1,2,\ldots,n-1).$$

Putting $\alpha = \frac{2m+1}{2n}\pi$, we can write

$$c_k^{2(m-n)+1} = \exp\left[i\frac{(2k+1)\pi(2m-2n+1)}{2n}\right]$$

$$= \exp\left[i\frac{(2k+1)(2m+1)\pi}{2n}\right]\exp[-i(2k+1)\pi] = -e^{i(2k+1)\alpha}.$$

Thus

$$\operatorname*{Res}_{z=c_k}\frac{z^{2m}}{z^{2n}+1} = -\frac{1}{2n}e^{i(2k+1)\alpha} \qquad (k=0,1,2,\ldots,n-1).$$

In view of the identity (see Exercise 9, Sec. 8)

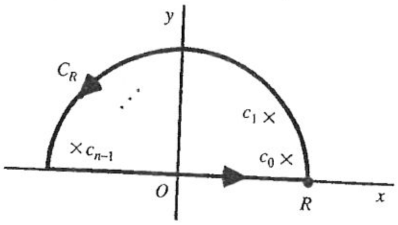
$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z} \qquad (z \ne 1),$$

then,

$$2\pi i\sum_{k=0}^{n-1}\operatorname*{Res}_{z=c_k}\frac{z^{2m}}{z^{2n}+1}=-\frac{\pi i}{n}e^{i\alpha}\sum_{k=0}^{n-1}(e^{i2\alpha})^k=-\frac{\pi i}{n}e^{i\alpha}\frac{1-e^{i2\alpha n}}{1-e^{i2\alpha}}\cdot\frac{e^{-i\alpha}}{e^{-i\alpha}}=-\frac{\pi i}{n}\cdot\frac{e^{i2\alpha n}-1}{e^{i\alpha}-e^{-i\alpha}}$$

$$=-\frac{\pi i}{n}\cdot\frac{e^{i(2m+1)\pi}-1}{e^{i\alpha}-e^{-i\alpha}}=\frac{\pi}{n}\cdot\frac{2i}{e^{i\alpha}-e^{-i\alpha}}=\frac{\pi}{n\sin\alpha}.$$

(c) Consider the path shown below, where $R>1$.



The residue theorem tells us that

$$\int_{-R}^{R}\frac{x^{2m}}{x^{2n}+1}dx+\int_{C_R}\frac{z^{2m}}{z^{2n}+1}dz=2\pi i\sum_{k=0}^{n-1}\operatorname*{Res}_{z=c_k}\frac{z^{2m}}{z^{2n}+1},$$

or

$$\int_{-R}^{R}\frac{x^{2m}}{x^{2n}+1}dx=\frac{\pi}{n\sin\alpha}-\int_{C_R}\frac{z^{2m}}{z^{2n}+1}dz.$$

Observe that if z is a point on C_R, then

$$|z^{2m}|=R^{2m}\quad\text{and}\quad|z^{2n}+1|\geq R^{2n}-1.$$

Consequently,

$$\left|\int_{C_R}\frac{z^{2m}}{z^{2n}+1}dz\right|\leq\frac{R^{2m}}{R^{2n}-1}\pi R\cdot\frac{R^{-2n}}{R^{-2n}}=\pi\frac{\dfrac{1}{R^{2(n-m)-1}}}{1-\dfrac{1}{R^{2n}}}\to 0;$$

and the desired integration formula follows.