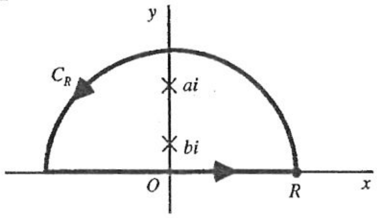
①

p. 275

1. The problem here is to evaluate the integral $\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}$, where $a > b > 0$. To do this, we introduce the function $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$, whose singularities ai and bi lie inside the simple closed contour shown below, where $R > a$. The other singularities are, of course, in the lower half plane.



According to the residue theorem,

$$\int_{-R}^{R} \frac{e^{ix}\,dx}{(x^2+a^2)(x^2+b^2)} + \int_{C_R} f(z)e^{iz}dz = 2\pi i(B_1+B_2),$$

where

$$B_1 = \operatorname*{Res}_{z=ai}[f(z)e^{iz}] = \left[\frac{e^{iz}}{(z+ai)(z^2+b^2)}\right]_{z=ai} = \frac{e^{-a}}{2a(b^2-a^2)i}$$

and

$$B_2 = \operatorname*{Res}_{z=bi}[f(z)e^{iz}] = \left[\frac{e^{iz}}{(z^2+a^2)(z+bi)}\right]_{z=bi} = \frac{e^{-b}}{2b(a^2-b^2)i}.$$

That is,

$$\int_{-R}^{R} \frac{e^{ix}\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) - \int_{C_R} f(z)e^{iz}dz,$$

or

$$\int_{-R}^{R} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) - \operatorname{Re}\int_{C_R} f(z)e^{iz}dz.$$

Now, if z is a point on C_R,

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{1}{(R^2-a^2)(R^2-b^2)}$$

and $|e^{iz}| = e^{-y} \le 1$. Hence

$$\left|\operatorname{Re}\int_{C_R} f(z)e^{iz}dz\right| \le \left|\int_{C_R} f(z)e^{iz}dz\right| \le M_R \pi R = \frac{\pi R}{(R^2-a^2)(R^2-b^2)} \to 0 \text{ as } R \to \infty.$$

So it follows that

$$\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) \qquad (a > b > 0).$$

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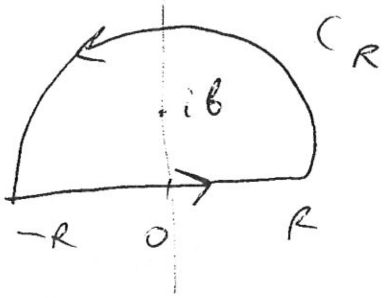
$$I = \int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx \qquad (a>0,\ b>0)$$

$$I_0 = \int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+b^2)^2}\,dx = 2I$$

$$f(z) = \frac{e^{iaz}}{(z^2+b^2)^2}$$

pole of second order



$$\left(\int_{-R}^{R} + \int_{C_R}\right) f(z)\,dz = 2\pi i \operatorname*{Res}_{z=ib} f(z)$$

$$= 2\pi i \operatorname*{Res}_{z=ib} \frac{e^{iaz}}{(z+ib)^2(z-ib)^2} = 2\pi i \left(\frac{e^{iaz}}{(z+ib)^2}\right)'\Bigg|_{z=ib}$$

$$= 2\pi i \, \frac{ia e^{iaz}(z+ib)^2 - e^{iaz}\,2(z+ib)}{(z+ib)^4}\Bigg|_{z=ib}$$

$$= 2\pi i \, \frac{e^{-ab}\left[ia(2ib)^2 - 2\cdot 2ib\right]}{(2ib)^4} = -\frac{2\pi i\cdot 2}{(2ib)^3}[-1+ab]e^{-ab}$$

$$= \frac{\pi}{2} e^{-ab} \frac{1+ab}{b^3}$$

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0 \quad \text{by Jordan's Lemma.}$$

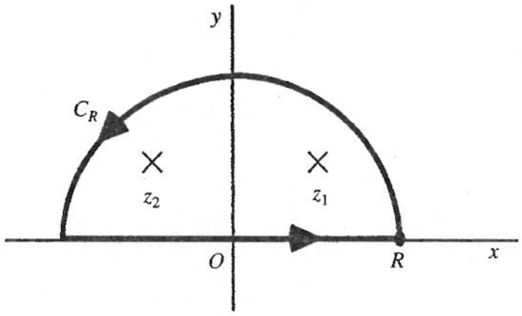
$$\Rightarrow I = \int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{I_0}{2}$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} e^{-ab} \frac{1+ab}{b^3} = \frac{\pi}{4b^3} e^{-ab}(1+ab)$$

6. The integral to be evaluated is $\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4}\,dx$, where $a>0$. We define the function $f(z)=\frac{z^3}{z^4+4}$; and, by computing the fourth roots of -4, we find that the singularities

$$z_1=\sqrt{2}e^{i\pi/4}=1+i \quad \text{and} \quad z_2=\sqrt{2}e^{i3\pi/4}=\sqrt{2}e^{i\pi/4}e^{i\pi/2}=(1+i)i=-1+i$$

both lie inside the simple closed contour shown below, where $R>\sqrt{2}$. The other two singularities lie below the real axis.



The residue theorem and the method of Theorem 2 in Sec. 76 for finding residues at simple poles tell us that

$$\int_{-R}^{R} \frac{x^3 e^{iax}}{x^4+4}\,dx + \int_{C_R} f(z)e^{iaz}\,dz = 2\pi i(B_1+B_2),$$

where

$$B_1 = \operatorname*{Res}_{z=z_1} \frac{z^3 e^{iaz}}{z^4+4} = \frac{z_1^3 e^{iaz_1}}{4z_1^3} = \frac{e^{iaz_1}}{4} = \frac{e^{ia(1+i)}}{4} = \frac{e^{-a}e^{ia}}{4}$$

and

$$B_2 = \operatorname*{Res}_{z=z_2} \frac{z^3 e^{iaz}}{z^4+4} = \frac{z_2^3 e^{iaz_2}}{4z_2^3} = \frac{e^{iaz_2}}{4} = \frac{e^{ia(-1+i)}}{4} = \frac{e^{-a}e^{-ia}}{4}.$$

Since

$$2\pi i(B_1 + B_2) = \pi i e^{-a}\left(\frac{e^{ia}+e^{-ia}}{2}\right) = i\pi e^{-a}\cos a,$$

we are now able to write

$$\int_{-R}^{R} \frac{x^3 \sin ax}{x^4+4}\,dx = \pi e^{-a}\cos a - \operatorname{Im}\int_{C_R} f(z)e^{iaz}\,dz.$$

Furthermore, if z is a point on C_R, then

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{R^3}{R^4-4} \to 0 \text{ as } R \to \infty;$$

and this means that

$$\left|\operatorname{Im}\int_{C_R} f(z)e^{iaz}\,dz\right| \le \left|\int_{C_R} f(z)e^{iaz}\,dz\right| \to 0 \text{ as } R \to \infty,$$

according to limit (1), Sec. 74. Finally, then,

$$\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4}\,dx = \pi e^{-a}\cos a \qquad (a>0).$$

10. To find the Cauchy principal value of the improper integral $\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}dx$, we shall use the function $f(z) = \frac{z+1}{z^2+4z+5} = \frac{z+1}{(z-z_1)(z-\overline{z}_1)}$, where $z_1 = -2+i$, and $\overline{z}_1 = -2-1$, and the same simple closed contour as in Exercise 9. In this case,

$$\int_{-R}^{R} \frac{(x+1)e^{ix}\,dx}{x^2+4x+5} + \int_{C_R} f(z)e^{iz}dz = 2\pi iB,$$

where

$$B = \operatorname*{Res}_{z=z_1}\left[\frac{(z+1)e^{iz}}{(z-z_1)(z-\overline{z}_1)}\right] = \frac{(z_1+1)e^{iz_1}}{(z-\overline{z}_1)} = \frac{(-1+i)e^{-2i}}{2ei}.$$

Thus

$$\int_{-R}^{R} \frac{(x+1)\cos x}{x^2+4x+5}dx = \operatorname{Re}(2\pi iB) - \int_{C_R} f(z)e^{iz},$$

or

$$\int_{-R}^{R} \frac{(x+1)\cos x}{x^2+4x+5}dx = \frac{\pi}{e}(\sin 2 - \cos 2) - \int_{C_R} f(z)e^{iz}dz.$$

Finally, we observe that if z is a point on C_R, then

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{R+1}{(R-|z_1|)(R-|\overline{z}_1|)} = \frac{R+1}{(R-\sqrt{5})^2} \to 0 \text{ as } R \to \infty.$$

The theorem in Sec. 81 then tells us that

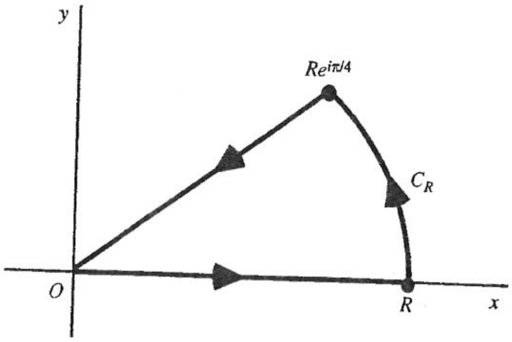
$$\left|\operatorname{Re}\int_{C_R} f(z)e^{iz}dz\right| \le \left|\int_{C_R} f(z)e^{iz}dz\right| \to 0 \text{ as } R \to \infty,$$

and so

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}dx = \frac{\pi}{e}(\sin 2 - \cos 2).$$

(optional) ⑥

12. (a) Since the function $f(z) = \exp(iz^2)$ is entire, the Cauchy-Goursat theorem tells us that its integral around the positively oriented boundary of the sector $0 \le r \le R,\ 0 \le \theta \le \pi/4$ has value zero. The closed path is shown below.



A parametric representation of the horizontal line segment from the origin to the point R is $z = x\ (0 \le x \le R)$, and a representation for the segment from the origin to the point $Re^{i\pi/4}$ is $z = re^{i\pi/4}\ (0 \le r \le R)$. Thus

$$\int_0^R e^{ix^2}dx + \int_{C_R} e^{iz^2}dz - e^{i\pi/4}\int_0^R e^{-r^2}dr = 0,$$

or

$$\int_0^R e^{ix^2}dx = e^{i\pi/4}\int_0^R e^{-r^2}dr - \int_{C_R} e^{iz^2}dz.$$

By equating real parts and then imaginary parts on each side of this last equation, we see that

$$\int_0^R \cos(x^2)dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}dr - \operatorname{Re}\int_{C_R} e^{iz^2}dz$$

and

$$\int_0^R \sin(x^2)dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}dr - \operatorname{Im}\int_{C_R} e^{iz^2}dz.$$

(b) A parametric representation for the arc C_R is $z = Re^{i\theta}\ (0 \le \theta \le \pi/4)$. Hence

$$\int_{C_R} e^{iz^2}dz = \int_0^{\pi/4} e^{iR^2e^{i2\theta}}Rie^{i\theta}d\theta = iR\int_0^{\pi/4} e^{-R^2\sin 2\theta}e^{iR^2\cos 2\theta}e^{i\theta}d\theta.$$

Since $\left|e^{iR^2\cos 2\theta}\right| = 1$ and $\left|e^{i\theta}\right| = 1$, it follows that

$$\left|\int_{C_R} e^{iz^2}dz\right| \le R\int_0^{\pi/4} e^{-R^2\sin 2\theta}d\theta.$$

Then, by making the substitution $\phi = 2\theta$ in this last integral and referring to the form (2), Sec. 81, of Jordan's inequality, we find that

$$\left|\int_{C_R} e^{iz^2}dz\right| \le \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\phi}d\phi \le \frac{R}{2}\cdot\frac{\pi}{2R^2} = \frac{\pi}{4R} \to 0 \text{ as } R \to \infty.$$

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(c) In view of the result in part (b) and the integration formula

$$\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2},$$

it follows from the last two equations in part (a) that

$$\int_0^{\infty} \cos(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} \quad \text{and} \quad \int_0^{\infty} \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

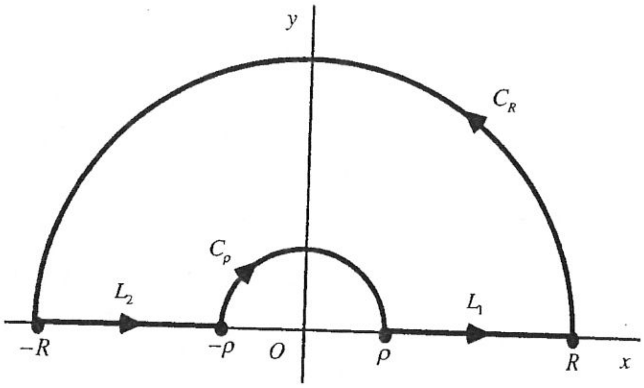
p. 286

8

1. The main problem here is to derive the integration formula

$$\int_0^\infty \frac{\cos(ax)-\cos(bx)}{x^2}\,dx = \frac{\pi}{2}(b-a) \qquad (a \ge 0,\, b \ge 0),$$

using the indented contour shown below.



Applying the Cauchy-Goursat theorem to the function

$$f(z) = \frac{e^{iaz}-e^{ibz}}{z^2},$$

we have

$$\int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C_\rho} f(z)\,dz = 0,$$

or

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = -\int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Since L_1 and $-L_2$ have parametric representations

$$L_1 : z = re^{i0} = r \; (\rho \le r \le R) \quad \text{and} \quad -L_2 : z = re^{i\pi} = -r \; (\rho \le r \le R),$$

we can see that

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = \int_{L_1} f(z)\,dz - \int_{-L_2} f(z)\,dz = \int_\rho^R \frac{e^{iar}-e^{ibr}}{r^2}\,dr + \int_\rho^R \frac{e^{-iar}-e^{-ibr}}{r^2}\,dr$$

$$= \int_\rho^R \frac{(e^{iar}+e^{-iar})-(e^{ibr}+e^{-ibr})}{r^2}\,dr = 2\int_\rho^R \frac{\cos(ar)-\cos(br)}{r^2}\,dr.$$

Thus

$$2\int_{\rho}^{R} \frac{\cos(ar)-\cos(br)}{r^2}\,dr = -\int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

In order to find the limit of the first integral on the right here as $\rho \to 0$, we write

$$f(z) = \frac{1}{z^2}\left[\left(1+\frac{iaz}{1!}+\frac{(iaz)^2}{2!}+\frac{(iaz)^3}{3!}+\cdots\right)-\left(1+\frac{ibz}{1!}+\frac{(ibz)^2}{2!}+\frac{(ibz)^3}{3!}+\cdots\right)\right]$$

$$= \frac{i(a-b)}{z}+\cdots \quad (0<|z|<\infty).$$

From this we see that $z=0$ is a simple pole of $f(z)$, with residue $B_0 = i(a-b)$. Thus

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)\,dz = -B_0\pi i = -i(a-b)\pi i = \pi(a-b).$$

As for the limit of the value of the second integral as $R\to\infty$, we note that if z is a point on C_R, then

$$f(z) \le \frac{|e^{iaz}|+|e^{ibz}|}{|z|^2} = \frac{e^{-ay}+e^{-by}}{R^2} \le \frac{1+1}{R^2} = \frac{2}{R^2}.$$

Consequently,

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{2}{R^2}\pi R = \frac{2\pi}{R} \to 0 \text{ as } R\to\infty.$$

It is now clear that letting $\rho\to 0$ and $R\to\infty$ yields

$$2\int_{0}^{\infty} \frac{\cos(ar)-\cos(br)}{r^2}\,dr = \pi(b-a).$$

This is the desired integration formula, with the variable of integration r instead of x. Observe that when $a=0$ and $b=2$, that result becomes

$$\int_{0}^{\infty} \frac{1-\cos(2x)}{x^2}\,dx = \pi.$$

But $\cos(2x) = 1-2\sin^2 x$, and we arrive at

$$\int_{0}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

HW 1 Solutions

①

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4. Let us use the function

$$f(z) = \frac{(\log z)^2}{z^2+1} \qquad \left(|z|>0,\ -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

and the contour in Exercise 2 to show that

$$\int_0^\infty \frac{(\ln x)^2}{x^2+1}dx = \frac{\pi^3}{8} \quad \text{and} \quad \int_0^\infty \frac{\ln x}{x^2+1}dx = 0.$$

Integrating $f(z)$ around the closed path shown in Exercise 2, we have

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = 2\pi i \operatorname*{Res}_{z=i} f(z) - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz.$$

Since

$$f(z) = \frac{\phi(z)}{z-i} \quad \text{where} \quad \phi(z) = \frac{(\log z)^2}{z+i},$$

the point $z=i$ is a simple pole of $f(z)$ and the residue is

$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = \frac{(\log i)^2}{2i} = \frac{(\ln 1 + i\pi/2)^2}{2i} = -\frac{\pi^2}{8i}.$$

Also, the parametric representations

$$L_1: z = re^{i0} = r\ (\rho \le r \le R) \quad \text{and} \quad -L_2: z = re^{i\pi} = -r\ (\rho \le r \le R)$$

enable us to write

$$\int_{L_1} f(z)dz = \int_\rho^R \frac{(\ln r)^2}{r^2+1}dr \quad \text{and} \quad \int_{L_2} f(z)dz = \int_\rho^R \frac{(\ln r + i\pi)^2}{r^2+1}dr.$$

Since

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = 2\int_\rho^R \frac{(\ln r)^2}{r^2+1}dr - \pi^2\int_\rho^R \frac{dr}{r^2+1} + 2\pi i \int_\rho^R \frac{\ln r}{r^2+1}dr,$$

then,

$$2\int_\rho^R \frac{(\ln r)^2}{r^2+1}dr - \pi^2\int_\rho^R \frac{dr}{r^2+1} + 2\pi i \int_\rho^R \frac{\ln r}{r^2+1}dr = -\frac{\pi^3}{4} - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz.$$

Equating real parts on each side of this equation, we have

$$2\int_\rho^R \frac{(\ln r)^2}{r^2+1}dr - \pi^2\int_\rho^R \frac{dr}{r^2+1} = -\frac{\pi^3}{4} - \operatorname{Re}\int_{C_\rho} f(z)dz - \operatorname{Re}\int_{C_R} f(z)dz;$$

and equating imaginary parts yields

$$2\pi\int_{\rho}^{R}\frac{\ln r}{r^2+1}dr = \operatorname{Im}\int_{C_\rho} f(z)dz - \operatorname{Im}\int_{C_R} f(z)dz.$$

It is straightforward to show that

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)dz = 0 \quad \text{and} \quad \lim_{R\to\infty}\int_{C_R} f(z)dz = 0.$$

Hence

$$2\int_0^\infty \frac{(\ln r)^2}{r^2+1}dr - \pi^2\int_0^\infty \frac{dr}{r^2+1} = -\frac{\pi^3}{4}$$

and

$$2\pi\int_0^\infty \frac{\ln r}{r^2+1}dr = 0.$$

Finally, inasmuch as (see Exercise 1, Sec. 79),

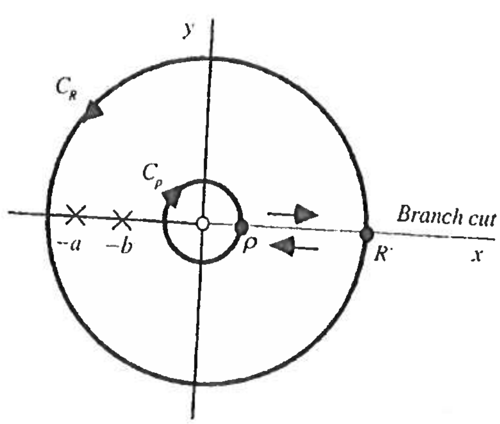
$$\int_0^\infty \frac{dr}{r^2+1} = \frac{\pi}{2},$$

we arrive at the desired integration formulas.

5. Here we evaluate the integral $\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}dx$, where $a>b>0$. We consider the function

$$f(z) = \frac{z^{1/3}}{(z+a)(z+b)} = \frac{\exp\left(\frac{1}{3}\log z\right)}{(z+a)(z+b)} \qquad (|z|>0, 0<\arg z<2\pi)$$

and the simple closed contour shown below, which is similar to the one used in Sec. 77. The numbers ρ and R are small and large enough, respectively, so that the points $z=-a$ and $z=-b$ are between the circles.



(3)

A parametric representation for the upper edge of the branch cut from ρ to R is $z = re^{i0}$ $(\rho \le r \le R)$, and so the value of the integral of f along that edge is

$$\int_\rho^R \frac{\exp\left[\frac{1}{3}(\ln r + i0)\right]}{(r+a)(r+b)}dr = \int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr.$$

A representation for the lower edge from ρ to is R is $z = re^{i2\pi}$ $(\rho \le r \le R)$. Hence the value of the integral of f along that edge from R to ρ is

$$-\int_\rho^R \frac{\exp\left[\frac{1}{3}(\ln r + i2\pi)\right]}{(r+a)(r+b)}dr = -e^{i2\pi/3}\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr.$$

According to the residue theorem, then,

$$\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr + \int_{C_R} f(z)dz - e^{i2\pi/3}\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr + \int_{C_\rho} f(z)dz = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname*{Res}_{z=-a} f(z) = \frac{\exp\left[\frac{1}{3}\log(-a)\right]}{-a+b} = -\frac{\exp\left[\frac{1}{3}(\ln a + i\pi)\right]}{a-b} = -\frac{e^{i\pi/3}\sqrt[3]{a}}{a-b}$$

and

$$B_2 = \operatorname*{Res}_{z=-b} f(z) = \frac{\exp\left[\frac{1}{3}\log(-b)\right]}{-b+a} = \frac{\exp\left[\frac{1}{3}(\ln b + i\pi)\right]}{-b+a} = \frac{e^{i\pi/3}\sqrt[3]{b}}{a-b}.$$

Consequently,

$$\left(1 - e^{i2\pi/3}\right)\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr = -\frac{2\pi i e^{i\pi/3}(\sqrt[3]{a} - \sqrt[3]{b})}{a-b} - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz.$$

Now

$$\left|\int_{C_\rho} f(z)dz\right| \le \frac{\sqrt[3]{\rho}}{(a-\rho)(b-\rho)}2\pi\rho = \frac{2\pi\sqrt[3]{\rho}\,\rho}{(a-\rho)(b-\rho)} \to 0 \text{ as } \rho \to 0$$

and

$$\left|\int_{C_R} f(z)dz\right| \le \frac{\sqrt[3]{R}}{(R-a)(R-b)}2\pi R = \frac{2\pi R^2}{(R-a)(R-b)} \cdot \frac{1}{\sqrt[3]{R^2}} \to 0 \text{ as } R \to \infty.$$

4

Hence

$$\int_0^{\infty}\frac{\sqrt[3]{r}}{(r+a)(r+b)}dr=-\frac{2\pi i e^{i\pi/3}(\sqrt[3]{a}-\sqrt[3]{b})}{\left(1-e^{i2\pi/3}\right)(a-b)}\cdot\frac{e^{-i\pi/3}}{e^{-i\pi/3}}=\frac{2\pi i(\sqrt[3]{a}-\sqrt[3]{b})}{\left(e^{i\pi/3}-e^{-i\pi/3}\right)(a-b)}$$

$$=\frac{\pi(\sqrt[3]{a}-\sqrt[3]{b})}{\sin(\pi/3)(a-b)}=\frac{\pi(\sqrt[3]{a}-\sqrt[3]{b})}{\frac{\sqrt{3}}{2}(a-b)}=\frac{2\pi}{\sqrt{3}}\cdot\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b}.$$

Replacing the variable of integration r here by x, we have the desired result:

$$\int_0^{\infty}\frac{\sqrt[3]{x}}{(x+a)(x+b)}dx=\frac{2\pi}{\sqrt{3}}\cdot\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b}\qquad (a>b>0).$$

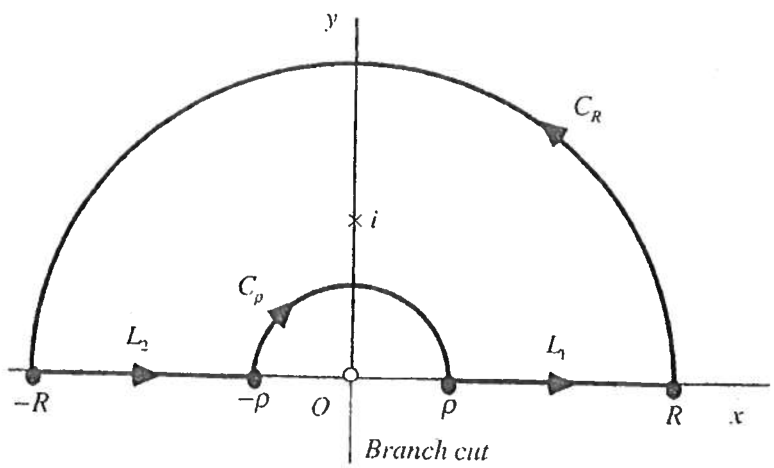
(optional)

6. (a) Let us first use the branch

$$f(z)=\frac{z^{-1/2}}{z^2+1}=\frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2+1}\qquad \left(|z|>0,-\frac{\pi}{2}<\arg z<\frac{3\pi}{2}\right)$$

and the indented path shown below to evaluate the improper integral

$$\int_0^{\infty}\frac{dx}{\sqrt{x}(x^2+1)}.$$



Cauchy's residue theorem tells us that

$$\int_{L_1}f(z)dz+\int_{C_R}f(z)dz+\int_{L_2}f(z)dz+\int_{C_\rho}f(z)dz=2\pi i\operatorname*{Res}_{z=i}f(z),$$

or

$$\int_{L_1}f(z)dz+\int_{L_2}f(z)dz=2\pi i\operatorname*{Res}_{z=i}f(z)-\int_{C_\rho}f(z)dz-\int_{C_R}f(z)dz.$$

5

Since

$$L_1: z = re^{i0} = r\ (\rho \le r \le R) \quad \text{and} \quad -L_2: z = re^{i\pi} = -r\ (\rho \le r \le R),$$

we may write

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = \int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)} - i\int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)} = (1-i)\int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)}.$$

Thus

$$(1-i)\int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)} = 2\pi i \operatorname*{Res}_{z=i} f(z) - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz.$$

Now the point $z = i$ is evidently a simple pole of $f(z)$, with residue

$$\operatorname*{Res}_{z=i} f(z) = \left[\frac{z^{-1/2}}{z+i}\right]_{z=i} = \frac{\exp\left[-\frac{1}{2}\log i\right]}{2i} = \frac{\exp\left[-\frac{1}{2}\left(\ln 1 + i\frac{\pi}{2}\right)\right]}{2i} = \frac{e^{-i\pi/4}}{2i} = \frac{1}{2i}\left(\frac{1-i}{\sqrt{2}}\right).$$

Furthermore,

$$\left|\int_{C_\rho} f(z)dz\right| \le \frac{\pi\rho}{\sqrt{\rho}(1-\rho^2)} = \frac{\pi\sqrt{\rho}}{1-\rho^2} \to 0 \text{ as } \rho \to 0$$

and

$$\left|\int_{C_R} f(z)dz\right| \le \frac{\pi\sqrt{R}}{(R^2-1)} = \frac{\pi}{\sqrt{R}\left(R-\dfrac{1}{R}\right)} \to 0 \text{ as } R \to \infty.$$

Finally, then, we have

$$(1-i)\int_0^\infty \frac{dr}{\sqrt{r}(r^2+1)} = \frac{\pi(1-i)}{\sqrt{2}},$$

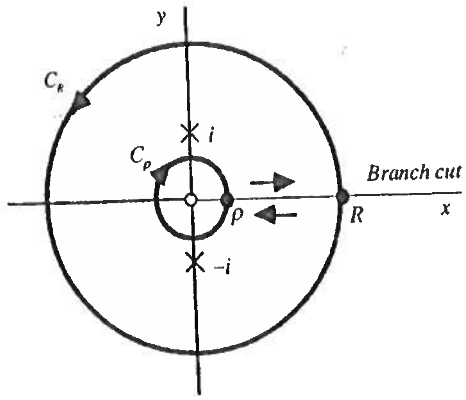
which is the same as

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

(b) To evaluate the improper integral $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)}$, we now use the branch

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2+1} \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

and the simple closed contour shown in the figure below, which is similar to Fig. 103 in Sec. 84. We stipulate that $\rho < 1$ and $R > 1$, so that the singularities $z = \pm i$ are between C_ρ and C_R.



Since a parametric representation for the upper edge of the branch cut from ρ to R is $z = re^{i0}$ $(\rho \le r \le R)$, the value of the integral of f along that edge is

$$\int_\rho^R \frac{\exp\left[-\frac{1}{2}(\ln r + i0)\right]}{r^2+1}\,dr = \int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr.$$

A representation for the lower edge from ρ to is R is $(\rho \le r \le R)$, and so the value of the integral of f along that edge from R to ρ is

$$-\int_\rho^R \frac{\exp\left[-\frac{1}{2}(\ln r + i2\pi)\right]}{r^2+1}\,dr = -e^{-i\pi}\int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr = \int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr.$$

Hence, by the residue theorem,

$$\int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr + \int_{C_R} f(z)\,dz + \int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr + \int_{C_\rho} f(z)\,dz = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname*{Res}_{z=i} f(z) = \left[\frac{z^{-1/2}}{z+i}\right]_{z=i} = \frac{\exp\left[-\frac{1}{2}\log i\right]}{2i} = \frac{\exp\left[-\frac{1}{2}\left(\ln 1 + i\frac{\pi}{2}\right)\right]}{2i} = \frac{e^{-i\pi/4}}{2i}$$

and

$$B_2 = \operatorname*{Res}_{z=-i} f(z) = \left[\frac{z^{-1/2}}{z-i}\right]_{z=-i} = \frac{\exp\left[-\frac{1}{2}\log(-i)\right]}{-2i} = \frac{\exp\left[-\frac{1}{2}\left(\ln 1 + i\frac{3\pi}{2}\right)\right]}{-2i} = -\frac{e^{-i3\pi/4}}{2i}.$$

That is,

$$2\int_{\rho}^{R}\frac{1}{\sqrt{r}(r^2+1)}dr=\pi(e^{-i\pi/4}-e^{-i3\pi/4})-\int_{C_\rho}f(z)dz-\int_{C_R}f(z)dz.$$

Since

$$\left|\int_{C_\rho}f(z)dz\right|\le\frac{2\pi\rho}{\sqrt{\rho}(1-\rho^2)}=\frac{2\pi\sqrt{\rho}}{1-\rho^2}\to 0 \text{ as } \rho\to 0$$

and

$$\left|\int_{C_R}f(z)dz\right|\le\frac{2\pi R}{\sqrt{R}(R^2-1)}=\frac{2\pi}{\sqrt{R}\left(R-\frac{1}{R}\right)}\to 0 \text{ as } R\to\infty,$$

we now find that

$$\int_0^\infty\frac{1}{\sqrt{r}(r^2+1)}dr=\pi\frac{e^{-i\pi/4}-e^{-i3\pi/4}}{2}=\pi\frac{e^{-i\pi/4}+e^{-i3\pi/4}e^{i\pi}}{2}$$

$$=\pi\frac{e^{i\pi/4}+e^{-i\pi/4}}{2}=\pi\cos\left(\frac{\pi}{4}\right)=\frac{\pi}{\sqrt{2}}.$$

When x, instead of r, is used as the variable of integration here, we have the desired result:

$$\int_0^\infty\frac{dx}{\sqrt{x}(x^2+1)}=\frac{\pi}{\sqrt{2}}.$$

7 (optional) $(p>0,\ q>0)$

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\,dt$$

$$t=\frac{1}{x+1}\Rightarrow dt=-\frac{dx}{(x+1)^2}$$

$t=1\Rightarrow x=0$

$t=0\Rightarrow x=\infty$

$1-t=1-\frac{1}{x+1}=\frac{x}{x+1}$

$$\Rightarrow B(p,q)=-\int_\infty^0(1+x)^{-(p-1)}\left(\frac{x}{x+1}\right)^{q-1}\frac{dx}{(x+1)^2}$$

$$=\int_0^\infty(1+x)^{-p}x^{q-1}\,dx$$

$$B(p, 1-p) = \int_0^\infty (1+x)^{-p-(q-p)} x^{(q-1)} dx$$

$q = 1-p > 0$
$\Rightarrow 0 < p < 1$

$$= \int_0^\infty \frac{x^{q-1}}{1+x} dx = \int_0^\infty \frac{x^{-p}}{x+1} dx, \quad \text{~~where~~}$$

$\Rightarrow$ from example of Sect 84.

$$B(p, 1-p) = \frac{\pi}{\sin(p\pi)}, \quad 0 < p < 1$$

9

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1 (optional)

Write

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_C \frac{1}{5+4\left(\frac{z-z^{-1}}{2i}\right)} \cdot \frac{dz}{iz} = \int_C \frac{dz}{2z^2+5iz-2},$$

where C is the positively oriented unit circle $|z|=1$. The quadratic formula tells us that the singular points of the integrand on the far right here are $z=-i/2$ and $z=-2i$. The point $z=-2i$ is exterior to C; and the point $z=-i/2$ is a simple pole interior to C. Thus

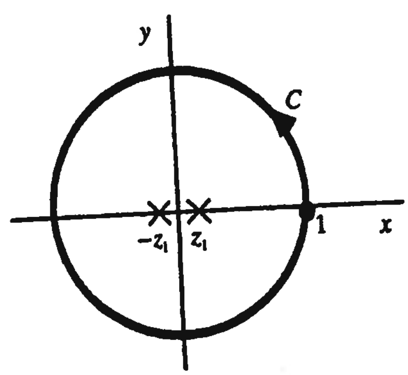
$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = 2\pi i \operatorname*{Res}_{z=-i/2}\left[\frac{1}{2z^2+5iz-2}\right] = 2\pi i\left[\frac{1}{4z+5i}\right]_{z=-i/2} = 2\pi i\left(\frac{1}{3i}\right) = \frac{2\pi}{3}.$$

2 (optional)

2. To evaluate the definite integral in question, write

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \int_C \frac{1}{1+\left(\frac{z-z^{-1}}{2i}\right)^2} \cdot \frac{dz}{iz} = \int_C \frac{4iz\,dz}{z^4-6z^2+1},$$

where C is the positively oriented unit circle $|z|=1$. This circle is shown below.



Solving the equation $(z^2)^2-6(z^2)+1=0$ for z^2 with the aid of the quadratic formula, we find that the zeros of the polynomial z^4-6z^2+1 are the numbers z such that $z^2=3\pm 2\sqrt{2}$. Those zeros are, then, $z=\pm\sqrt{3+2\sqrt{2}}$ and $z=\pm\sqrt{3-2\sqrt{2}}$. The first two of these zeros are exterior to the circle, and the second two are inside of it. So the singularities of the integrand in our contour integral are

$$z_1=\sqrt{3-2\sqrt{2}} \quad \text{and} \quad z_2=-z_1,$$

indicated in the figure. This means that

$$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}=2\pi i(B_1+B_2),$$

where

$$B_1=\operatorname*{Res}_{z=z_1}\frac{4iz}{z^4-6z^2+1}=\frac{4iz_1}{4z_1^3-12z_1}=\frac{i}{z_1^2-3}=\frac{i}{(3-2\sqrt{2})-3}=-\frac{i}{2\sqrt{2}}$$

and

$$B_2=\operatorname*{Res}_{z=-z_1}\frac{4iz}{z^4-6z^2+1}=\frac{-4iz_1}{-4z_1^3+12z_1}=\frac{i}{z_1^2-3}=-\frac{i}{2\sqrt{2}}.$$

Since

$$2\pi i(B_1+B_2)=2\pi i\left(-\frac{i}{\sqrt{2}}\right)=\frac{2\pi}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\sqrt{2}\pi,$$

the desired result is

$$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}=\sqrt{2}\pi.$$

7 (optional)

Let C be the positively oriented unit circle $|z|=1$. In view of the binomial formula (Sec. 3)

$$\int_0^{\pi}\sin^{2n}\theta\,d\theta=\frac{1}{2}\int_{-\pi}^{\pi}\sin^{2n}\theta\,d\theta=\frac{1}{2}\int_C\left(\frac{z-z^{-1}}{2i}\right)^{2n}\frac{dz}{iz}=\frac{1}{2^{2n+1}(-1)^n i}\int_C\frac{(z-z^{-1})^{2n}}{z}dz$$

$$=\frac{1}{2^{2n+1}(-1)^n i}\int_C\sum_{k=0}^{n}\binom{2n}{k}z^{2n-k}(-z^{-1})^k z^{-1}dz$$

$$=\frac{1}{2^{2n+1}(-1)^n i}\sum_{k=0}^{n}\binom{2n}{k}(-1)^k\int_C z^{2n-2k-1}\,dz.$$

Now each of these last integrals has value zero except when $k = n$:

$$\int_C z^{-1} dz = 2\pi i.$$

Consequently,

$$\int_0^{\pi} \sin^{2n} \theta \, d\theta = \frac{1}{2^{2n+1}(-1)^n i} \cdot \frac{(2n)!(-1)^n 2\pi i}{(n!)^2} = \frac{(2n)!}{2^{2n}(n!)^2} \pi.$$

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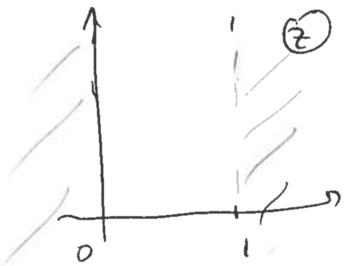
$w = u + iv$

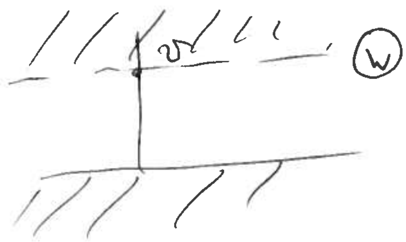
① $W = iz = e^{i\frac{\pi}{2}}z$

$z = re^{i\varphi} \Rightarrow w = re^{i(\varphi + \frac{\pi}{2})} \Rightarrow$ rotation at angle $\frac{\pi}{2}$



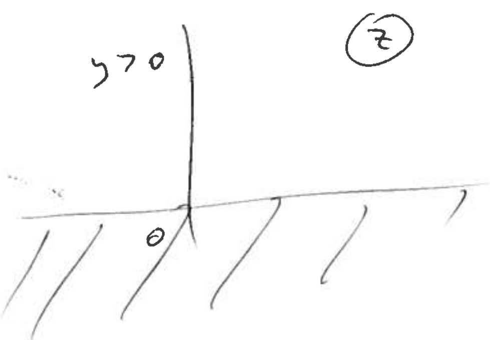
→ $x = 0$ maps into $-\infty < u < \infty$, $v = 0$

$x = \pi$ maps into $u + i$, $-\infty < u < \infty$

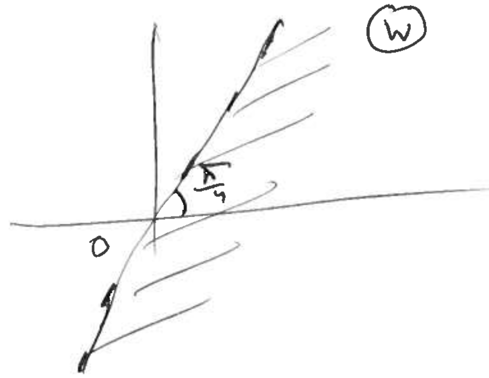


i.e. strip $0 < v < 1$

③ $w = (1+i)z = \sqrt{2}\,e^{i\frac{\pi}{4}}z \Rightarrow$ stretching by $\sqrt{2}$ and rotation by $\frac{\pi}{4}$



→



$\Rightarrow v > u$