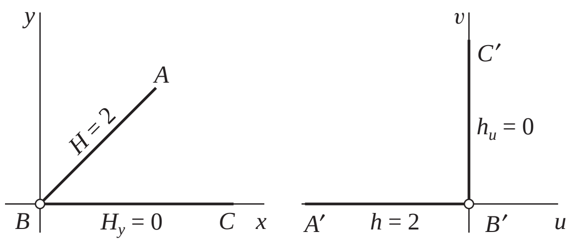

FIGURE 138

A boundary condition that is not of one of the two types mentioned in the theorem may be transformed into a condition that is substantially different from the original one (see Exercise 6). New boundary conditions for the transformed problem can be obtained for a particular transformation in any case. It is interesting to note that under a conformal transformation, the ratio of a directional derivative of H along a smooth arc C in the z plane to the directional derivative of h along the image curve Γ at the corresponding point in the w plane is $|f'(z)|$; usually, this ratio is not constant along a given arc. (See Exercise 10.)

EXERCISES

1. Use expression (5), Sec. 104, to find a harmonic conjugate of the harmonic function $u(x, y) = x^3 - 3xy^2$. Write the resulting analytic function in terms of the complex variable z.

2. Let $u(x, y)$ be harmonic in a simply connected domain D. By appealing to results in Secs. 104 and 52, show that its partial derivatives of all orders are continuous throughout that domain.

3. The transformation $w = \exp z$ maps the horizontal strip $0 < y < \pi$ onto the upper half plane $v > 0$, as shown in Fig. 6 of Appendix 2; and the function

$$h(u, v) = \text{Re}(w^2) = u^2 - v^2$$

is harmonic in that half plane. With the aid of the theorem in Sec. 105, show that the function $H(x, y) = e^{2x} \cos 2y$ is harmonic in the strip. Verify this result directly.

4. Under the transformation $w = \exp z$, the image of the segment $0 \leq y \leq \pi$ of the y axis is the semicircle $u^2 + v^2 = 1, v \geq 0$ (see Sec. 14). Also, the function

$$h(u, v) = \text{Re}\left(2 - w + \frac{1}{w}\right) = 2 - u + \frac{u}{u^2 + v^2}$$

is harmonic everywhere in the w plane except for the origin; and it assumes the value $h = 2$ on the semicircle. Write an explicit expression for the function $H(x, y)$ in the theorem of Sec. 106. Then illustrate the theorem by showing directly that $H = 2$ along the segment $0 \leq y \leq \pi$ of the y axis.

5. The transformation $w = z^2$ maps the positive x and y axes and the origin in the z plane onto the u axis in the w plane. Consider the harmonic function

$$h(u, v) = \text{Re}(e^{-w}) = e^{-u} \cos v,$$

and observe that its normal derivative h_v along the u axis is zero. Then illustrate the theorem in Sec. 106 when $f(z) = z^2$ by showing directly that the normal derivative of the function $H(x, y)$ defined in that theorem is zero along both positive axes in the z plane. (Note that the transformation $w = z^2$ is not conformal at the origin.)

6. Replace the function $h(u, v)$ in Exercise 5 by the harmonic function

$$h(u, v) = \text{Re}(-2iw + e^{-w}) = 2v + e^{-u} \cos v.$$

Then show that $h_v = 2$ along the u axis but that $H_y = 4x$ along the positive x axis and $H_x = 4y$ along the positive y axis. This illustrates how a condition of the type

$$\frac{dh}{dn} = h_0 \neq 0$$

is *not necessarily* transformed into a condition of the type $dH/dN = h_0$.

7. Show that if a function $H(x, y)$ is a solution of a Neumann problem (Sec. 105), then $H(x, y) + A$, where A is any real constant, is also a solution of that problem.

8. Suppose that an analytic function $w = f(z) = u(x, y) + iv(x, y)$ maps a domain D_z in the z plane onto a domain D_w in the w plane; and let a function $h(u, v)$, with continuous partial derivatives of the first and second order, be defined on D_w. Use the chain rule for partial derivatives to show that if $H(x, y) = h[u(x, y), v(x, y)]$, then

$$H_{xx}(x, y) + H_{yy}(x, y) = [h_{uu}(u, v) + h_{vv}(u, v)]\,|f'(z)|^2.$$

Conclude that the function $H(x, y)$ is harmonic in D_z when $h(u, v)$ is harmonic in D_w. This is an alternative proof of the theorem in Sec. 105, even when the domain D_w is multiply connected.

Suggestion: In the simplifications, it is important to note that since f is analytic, the Cauchy–Riemann equations $u_x = v_y$, $u_y = -v_x$ hold and that the functions u and v both satisfy Laplace's equation. Also, the continuity conditions on the derivatives of h ensure that $h_{vu} = h_{uv}$.

9. Let $p(u, v)$ be a function that has continuous partial derivatives of the first and second order and satisfies *Poisson's equation*

$$p_{uu}(u, v) + p_{vv}(u, v) = \Phi(u, v)$$

in a domain D_w of the w plane, where Φ is a prescribed function. Show how it follows from the identity obtained in Exercise 8 that if an analytic function

$$w = f(z) = u(x, y) + iv(x, y)$$

maps a domain D_z onto the domain D_w, then the function

$$P(x, y) = p[u(x, y), v(x, y)]$$

satisfies the Poisson equation

$$P_{xx}(x, y) + P_{yy}(x, y) = \Phi[u(x, y), v(x, y)]\,|f'(z)|^2$$

in D_z.

10. Suppose that $w = f(z) = u(x, y) + iv(x, y)$ is a conformal mapping of a smooth arc C onto a smooth arc Γ in the w plane. Let the function $h(u, v)$ be defined on Γ, and write

$$H(x, y) = h[u(x, y), v(x, y)].$$

(a) From calculus, we know that the x and y components of grad H are the partial derivatives H_x and H_y, respectively; likewise, grad h has components h_u and h_v. By applying the chain rule for partial derivatives and using the Cauchy–Riemann equations, show that if (x, y) is a point on C and (u, v) is its image on Γ, then

$$|\text{grad } H(x, y)| = |\text{grad } h(u, v)|\,|f'(z)|.$$

(b) Show that the angle from the arc C to grad H at a point (x, y) on C is equal to the angle from Γ to grad h at the image (u, v) of the point (x, y).

(c) Let s and σ denote distance along the arcs C and Γ, respectively; and let $\mathbf{t}$ and $\boldsymbol{\tau}$ denote unit tangent vectors at a point (x, y) on C and its image (u, v), in the direction of increasing distance. With the aid of the results in parts *(a)* and *(b)* and using the fact that

$$\frac{dH}{ds} = (\text{grad } H) \cdot \mathbf{t} \quad \text{and} \quad \frac{dh}{d\sigma} = (\text{grad } h) \cdot \boldsymbol{\tau},$$

show that the directional derivative along the arc Γ is transformed as follows:

$$\frac{dH}{ds} = \frac{dh}{d\sigma}\,|f'(z)|.$$