EXERCISES

1. State why the components of velocity can be obtained from the stream function by means of the equations

$$p(x, y) = \psi_y(x, y), \quad q(x, y) = -\psi_x(x, y).$$

2. At an interior point of a region of flow and under the conditions that we have assumed, the fluid pressure cannot be less than the pressure at all other points in a neighborhood of that point. Justify this statement with the aid of statements in Secs. 113, 114, and 54.

3. For the flow around a corner described in Example 1, Sec. 115, at what point of the region $x \geq 0$, $y \geq 0$ is the fluid pressure greatest?

4. Show that the speed of the fluid at points on the cylindrical surface in Example 2, Sec. 115, is $2A|\sin\theta|$ and also that the fluid pressure on the cylinder is greatest at the points $z = \pm 1$ and least at the points $z = \pm i$.

5. Write the complex potential for the flow around a cylinder $r = r_0$ when the velocity V at a point z approaches a real constant A as the point recedes from the cylinder.

6. Obtain the stream function $\psi = Ar^4 \sin 4\theta$ for a flow in the angular region

$$r \geq 0,\ 0 \leq \theta \leq \frac{\pi}{4}$$

that is shown in Fig. 160. Sketch a few of the streamlines in the interior of that region.

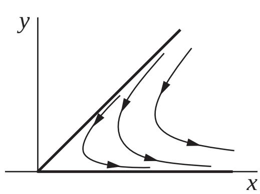

FIGURE 160

7. Obtain the complex potential $F = A \sin z$ for a flow inside the semi-infinite region

$$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2},\ y \geq 0$$

that is shown in Fig. 161. Write the equations of the streamlines.

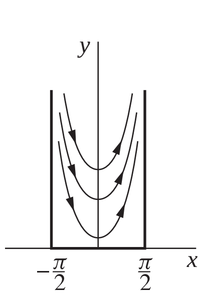

FIGURE 161

8. Show that if the velocity potential is $\phi = A \ln r\ (A > 0)$ for flow in the region $r \geq r_0$, then the streamlines are the half lines $\theta = c\ (r \geq r_0)$ and the rate of flow outward through each complete circle about the origin is $2\pi A$, corresponding to a source of that strength at the origin.

9. Obtain the complex potential

$$F = A\left(z^2 + \frac{1}{z^2}\right)$$

for a flow in the region $r \geq 1$, $0 \leq \theta \leq \pi/2$. Write expressions for V and ψ. Note how the speed $|V|$ varies along the boundary of the region, and verify that $\psi(x, y) = 0$ on the boundary.

10. Suppose that the flow at an infinite distance from the cylinder of unit radius in Example 2, Sec. 115, is uniform in a direction making an angle α with the x axis; that is,

$$\lim_{|z|\to\infty} V = Ae^{i\alpha} \qquad (A > 0).$$

Find the complex potential.

Ans. $F = A\left(ze^{-i\alpha} + \dfrac{1}{z}e^{i\alpha}\right).$

11. Write

$$z - 2 = r_1 \exp(i\theta_1), \quad z + 2 = r_2 \exp(i\theta_2),$$

and

$$(z^2 - 4)^{1/2} = \sqrt{r_1 r_2} \exp\left(i\frac{\theta_1 + \theta_2}{2}\right),$$

where

$$0 \leq \theta_1 < 2\pi \quad \text{and} \quad 0 \leq \theta_2 < 2\pi.$$

The function $(z^2 - 4)^{1/2}$ is then single-valued and analytic everywhere except on the branch cut consisting of the segment of the x axis joining the points $z = \pm 2$. We know, moreover, from Exercise 13, Sec. 92, that the transformation

$$z = w + \frac{1}{w}$$

maps the circle $|w| = 1$ onto the line segment from $z = -2$ to $z = 2$ and that it maps the domain outside the circle onto the rest of the z plane. Use all of the observations above to show that the inverse transformation, where $|w| > 1$ for every point not on the branch cut, can be written

$$w = \frac{1}{2}[z + (z^2 - 4)^{1/2}] = \frac{1}{4}\left(\sqrt{r_1}\exp\frac{i\theta_1}{2} + \sqrt{r_2}\exp\frac{i\theta_2}{2}\right)^2.$$

The transformation and this inverse establish a one to one correspondence between points in the two domains.

12. With the aid of the results found in Exercises 10 and 11, derive the expression

$$F = A[z\cos\alpha - i(z^2-4)^{1/2}\sin\alpha]$$

for the complex potential of the steady flow around a long plate whose width is 4 and whose cross section is the line segment joining the two points $z = \pm 2$ in Fig. 162, assuming that the velocity of the fluid at an infinite distance from the plate is $A\exp(i\alpha)$ where $A > 0$. The branch of $(z^2-4)^{1/2}$ that is used is the one described in Exercise 11.

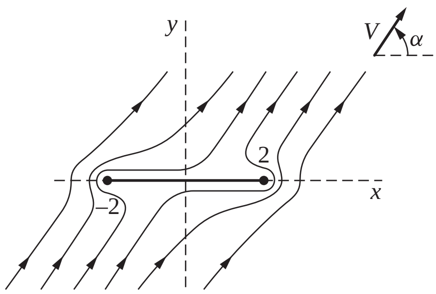

FIGURE 162

13. Show that if $\sin\alpha \neq 0$ in Exercise 12, then the speed of the fluid along the line segment joining the points $z = \pm 2$ is infinite at the ends and is equal to $A|\cos\alpha|$ at the midpoint.

14. For the sake of simplicity, suppose that $0 < \alpha \leq \pi/2$ in Exercise 12. Then show that the velocity of the fluid along the upper side of the line segment representing the plate in Fig. 162 is zero at the point $x = 2\cos\alpha$ and that the velocity along the lower side of the segment is zero at the point $x = -2\cos\alpha$.

15. A circle with its center at a point x_0 $(0 < x_0 < 1)$ on the x axis and passing through the point $z = -1$ is subjected to the transformation

$$w = z + \frac{1}{z}.$$

Individual nonzero points z can be mapped geometrically by adding the vectors representing

$$z = re^{i\theta} \quad \text{and} \quad \frac{1}{z} = \frac{1}{r}e^{-i\theta}.$$

Indicate by mapping some points that the image of the circle is a profile of the type shown in Fig. 163 and that points exterior to the circle map onto points exterior to the profile. This is a special case of the profile of a *Joukowski airfoil*. (See also Exercises 16 and 17 below.)

16. *(a)* Show that the mapping of the circle in Exercise 15 is conformal except at the point $z = -1$.

(b) Let the complex numbers

$$t = \lim_{\Delta z \to 0} \frac{\Delta z}{|\Delta z|} \quad \text{and} \quad \tau = \lim_{\Delta w \to 0} \frac{\Delta w}{|\Delta w|}$$

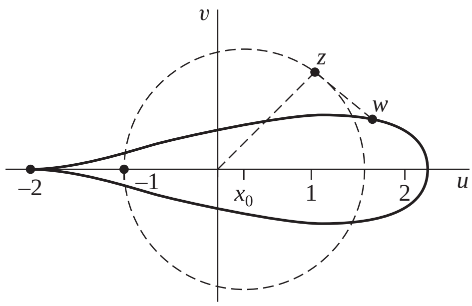

FIGURE 163

represent unit vectors tangent to a smooth directed arc at $z = -1$ and that arc's image, respectively, under the transformation

$$w = z + \frac{1}{z}.$$

Show that $\tau = -t^2$ and hence that the Joukowski profile in Fig. 163 has a cusp at the point $w = -2$, the angle between the tangents at the cusp being zero.

17. Find the complex potential for the flow around the airfoil in Exercise 15 when the velocity V of the fluid at an infinite distance from the origin is a real constant A. Recall that the inverse of the transformation

$$w = z + \frac{1}{z}$$

used in Exercise 15 is given, with z and w interchanged, in Exercise 11.

18. Note that under the transformation $w = e^z + z$, both halves, where $x \geq 0$ and $x \leq 0$, of the line $y = \pi$ are mapped onto the half line $v = \pi$ $(u \leq -1)$. Similarly, the line $y = -\pi$ is mapped onto the half line $v = -\pi$ $(u \leq -1)$; and the strip $-\pi \leq y \leq \pi$ is mapped onto the w plane. Also, note that the change of directions, $\arg(dw/dz)$, under this transformation approaches zero as x tends to $-\infty$. Show that the streamlines of a fluid flowing through the open channel formed by the half lines in the w plane (Fig. 164) are the images of the lines $y = c_2$ in the strip. These streamlines also represent the equipotential curves of the electrostatic field near the edge of a parallel-plate capacitor.

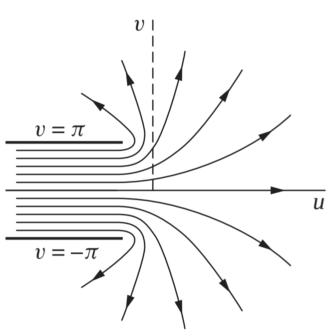

FIGURE 164