Test 1A solutions

1. Solve for the vector $\mathbf{x}$ in terms of $\mathbf{a}$ and $\mathbf{b}$:
$-2(3\mathrm{a} + 2\mathrm{x}) + 2\mathrm{a} + 5\mathbf{b} + 7\mathrm{x} = \mathbf{0}$.

$$-6\bar{a} - 4\bar{x} + 2\vec{a} + 5\vec{b} + 7\vec{x} = \vec{0}$$

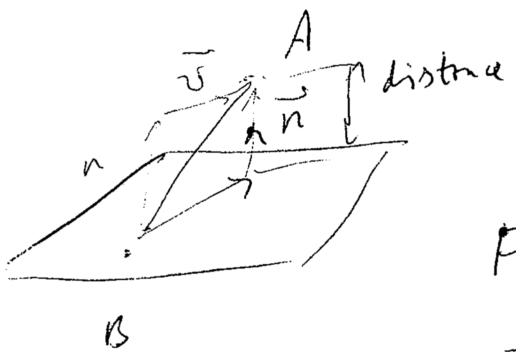
$$\Rightarrow \vec{x} = -\frac{1}{3}\left[-4\vec{a} + 5\vec{b}\right]$$

2. Find a vector projection of $\mathbf{x}$ onto $\mathbf{y}$. $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$.

$$\operatorname{proj}_{\vec{y}} \vec{x} = \frac{(\vec{x} \cdot \vec{y})}{\|\vec{y}\|^2} \vec{y} = \frac{(-2-2+12)}{4+1+16} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$

$$= \frac{8}{21} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{16}{21} \\ 8/21 \\ 32/21 \end{pmatrix}$$

3. Find a distance from the point $A = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ to the plane P which is defined by the general equation $x_1 + 3x_2 - x_3 = 2$.



$\vec{n} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

For A: $1 + 3\cdot(-3) - 1 = -9 \neq 2$

$\Rightarrow A \notin P$

choose $x_2 = x_3 = 0 \Rightarrow x_1 = 2$

$B = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \in P$

$\vec{v} = \overrightarrow{BA} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$

Projection of $\overrightarrow{BA}$ onto $\vec{n}$ defines $d(A, P)$:

$proj_{\vec{n}} \overrightarrow{BA} = \frac{\vec{n} \cdot \vec{v}}{\|\vec{n}\|^2} \vec{n} = \frac{-1-9-1}{1+9+1} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$

$d(A, P) = \| Proj_{\vec{n}} (\overrightarrow{BA}) \| = \sqrt{1+9+1} = \sqrt{11}$

4. (a) Find all solutions (if any) of the system of linear equations $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 0 & 1 & -3 \\ 2 & 1 & -1 \\ 1 & 3 & 0 \\ 0 & 2 & 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix}$.

(b) Is that system overdetermined or underdetermined?

(c) Is $\mathbf{b}$ in the span of the columns of A?

$$(A|\vec{b}) = \left(\begin{array}{ccc|c} 0 & 1 & -3 & 0 \\ 2 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 2 & 2 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & 2 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -5 & -1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & 2 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -5 & -1 & 1 \\ 0 & 2 & 2 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -16 & 1 \\ 0 & 0 & 8 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -1/16 \\ 0 & 0 & 0 & 7/2 \end{array}\right)$$

$\Rightarrow$ system is inconsistent.

(a) no solutions.

(b) System is overdetermined

(c) because there is no solutions

$\Rightarrow \vec{b} \notin \mathrm{span}(\vec{a}_1, \vec{a}_2, \vec{a}_3)$

5. Find all solutions (if any) of the system of linear equations $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} -2 & 3 & -1 \\ 4 & 2 & 10 \\ 6 & 7 & 19 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 6 \\ 13 \end{pmatrix}$.

$$(A|\vec{b}) = \left(\begin{array}{ccc|c} -2 & 3 & -1 & 1 \\ 4 & 2 & 10 & 6 \\ 6 & 7 & 19 & 13 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 3 & -1 & 1 \\ 0 & 8 & 8 & 8 \\ 0 & 16 & 16 & 16 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} -2 & 3 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 0 & -4 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_3 - free variable

$x_3 = \alpha$ - arbitrary constant

$\Rightarrow x_2 = -\alpha + 1$

$x_1 = -2\alpha + 1$

$$\vec{x} = \begin{pmatrix} -2\alpha + 1 \\ -\alpha + 1 \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

6. Find A-3B, where

$$A = \begin{pmatrix} -1 & 5 & 1 \\ 7 & 0 & 2 \\ 1 & 4 & 2 \end{pmatrix},$$

$$B = \begin{pmatrix} -2 & 3 & 1 \\ \frac{1}{3} & 1 & 1 \\ 1 & 0 & 4 \end{pmatrix}.$$

$$A - 3B = \begin{pmatrix} -1 & 5 & 1 \\ 7 & 0 & 2 \\ 1 & 4 & 2 \end{pmatrix} - 3\begin{pmatrix} -2 & 3 & 1 \\ \frac{1}{3} & 1 & 1 \\ 1 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -4 & -2 \\ 6 & -3 & -1 \\ -2 & 4 & -10 \end{pmatrix}$$

7. Find a matrix product AB if that product is defined, where

$$A = \begin{pmatrix} -1 & 7 & 2 \\ 2 & 1 & 5 \\ 2 & 1 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 4 & 5 & -2 \\ 1 & -2 & 3 \\ 2 & -3 & 1 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 7 & -25 & 25 \\ 19 & -7 & 4 \\ 7 & 11 & -2 \end{pmatrix}$$

8. Assume A is $n \times n$ matrices. Prove that $B = A + A^T$ is a symmetric matrix, i.e. $B^T = B$.

$$B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A$$

$$= A + A^T = B \Rightarrow B \text{ is symmetric } \blacksquare$$