Test 2A Math 221

1. Compute the inverse of the matrix $\begin{pmatrix} 2 & 1 & 4 \\ 1 & 0 & 2 \\ 0 & 3 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 & 6 & 1 \end{array}\right)$$

$$\underset{\text{(backward phase)}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -11 & -2 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 & 6 & 1 \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 6 & -11 & -2 \\ 1 & -2 & 0 \\ -3 & 6 & 1 \end{pmatrix}$$

2. Find an elementary matrix E such that $B = EA$.

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 3 & -1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 4 & 2 \\ 5 & 7 & 6 \\ 0 & 2 & 1 \end{pmatrix}.$$

we obtain B by row replacement:

$$row_2(B) = 2\, row_1(A) + row_2(A)$$

$$\Rightarrow E = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. Assume that S is a set of symmetric $n \times n$ real matrices.
(a) Show that S is a subspace of $\mathbb{R}^{n\times n}$ of all real $n \times n$ matrices.
(b) Find a basis for S.

(a) Assume $A^T = A$, $B^T = B$ – symmetric matrices $\in S$

$$(\alpha A + \beta B)^T = \alpha A^T + \beta B^T = \alpha A + \beta B \in S$$

for any A, B, α, β.

$\Rightarrow$ S is a subspace of $\mathbb{R}^{n\times n}$.

b) Symmetric matrix has a form:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & & \\ \vdots & & \ddots & \\ a_{1n} & & & a_{nn} \end{pmatrix}$$

$\Rightarrow$ need to span elements on main diag. and above it.

$$\Rightarrow \text{basis of } S = \left\{ \begin{pmatrix} 1 & 0 & \cdots \\ 0 & & \\ \vdots & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & & & \\ & & & \end{pmatrix}, \right.$$

$$\begin{array}{c} \\ i \\ \\ \end{array} \begin{pmatrix} 0 & & \\ & & 1 \\ & & \\ & & 0 \end{pmatrix}_{j}, \begin{pmatrix} \cancel{0} & & \\ & 0 & \\ & & \ddots \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \\ & & & \ddots \end{pmatrix}, \cdots \left. \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \\ & & & 1 \end{pmatrix} \right\}$$

– total $\frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$ vectors in basis.

4. Assume that $A_1, A_2, \ldots A_m$ are real $n \times n$ invertible matrices. Prove that $(A_1 A_2 \ldots A_m)^{-1} = A_m^{-1} \ldots A_2^{-1} A_1^{-1}$. $(*)$

① $m=1$ $(A_1)^{-1} = A_1^{-1}$

② Assume $(*)$ is true for m matrices:

$$(A_1 \ldots A_m)^{-1} = A_m^{-1} \ldots A_1^{-1} \equiv B^{-1}$$

Now calculate

$$(A_1 \ldots A_{m+1})^{-1} = (\underbrace{A_1 \ldots A_m}_{= B} A_m)^{-1}$$

$$= (B A_{m+1})^{-1} = A_{m+1}^{-1} B^{-1} = A_m^{-1} A_m^{-1} \ldots A_1^{-1}.$$

here we used property:

$$(B A_m)^{-1} = A_m^{-1} B^{-1}$$

because $B A_m A_m^{-1} B^{-1} = B B^{-1} = I$ and

similar $A_m^{-1} B^{-1} B A_m = A_m^{-1} A_m = I$.

$\blacksquare$

5. Find a basis of $Col(A)$, a basis of $Nul(A)$ and a basis of $Row(A)$. What is the rank of A?

Here $A = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 1 & -1 & 2 & 4 \\ 3 & 5 & 6 & 6 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 1 & -1 & 2 & 4 \\ 3 & 5 & 6 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 & 1 \\ 0 & -4 & 0 & 3 \\ 0 & -4 & 0 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 0 & -3/4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 1+\frac{9}{4} \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 2 & \frac{13}{4} \\ 0 & 1 & 0 & -3/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Row(A) = Span\Big\{ \underbrace{\left(1, 0, 2, \tfrac{13}{4}\right), \left(0, 1, 0, -\tfrac{3}{4}\right)}_{\text{basis of } Row(A)} \Big\}$$

pivot columns are 1, 2

$$\Rightarrow Col(A) = Span\Big\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}}_{\text{basis of } Col(A)} \Big\} \Rightarrow \boxed{rank(A) = 2}$$

$x_3 = \alpha$, $x_4 = \beta$ - free variables

$$\Rightarrow x_2 = \frac{3}{4}\beta \qquad \Rightarrow \vec{x} = \begin{pmatrix} -2\alpha - \frac{13}{4}\beta \\ 3/4\,\beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -\frac{13}{4} \\ 3/4 \\ 0 \\ 1 \end{pmatrix}$$

$$x_1 = -2\alpha - \frac{13}{4}\beta$$

$$\Rightarrow Null(A) = Span\Big\{ \underbrace{\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -13/4 \\ 3/4 \\ 0 \\ 1 \end{pmatrix}}_{\text{basis of } Null(A)} \Big\}$$

$p_1 \qquad p_2 \qquad p_3$

6. Determine if the following set of polynomials $\{2 + x + x^2, 1 - 2x + 2x^2, -2 + x + 3x^2\}$ form a basis in space $\mathbb{P}_2$ of all polynomial of order ≤ 2. Explain your answer.

Standard basis $E = \{1, x, x^2\}$

$$[p_1]_E = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$[p_2]_E = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$[p_3]_E = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

$$\det\left([p_1]_E \; [p_2]_E \; [p_3]_E \right) = \begin{vmatrix} 2 & 1 & -2 \\ 1 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= -26 \neq 0$$

$\Rightarrow \{[p_1]_E, [p_2]_E, [p_3]_E\}$ form a basis in $\mathbb{R}^3$

$\Rightarrow \{p_1, p_2, p_3\}$ form a basis in $\mathbb{P}_2$.

7. Determine if each of the following sets of vectors is a subspace of $\mathbb{R}^2$

(a)

$$S_1 = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ such that } 10x_1 - 3x_2 = 0 \right\}, \Rightarrow x_2 = \frac{10}{3} x_1$$

(b)

$$S_2 = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ such that } x_1 + 2x_2^2 = 0 \right\}.$$

(a) $\vec{x} = \begin{pmatrix} x_1 \\ \frac{10}{3} x_1 \end{pmatrix} \in S_1$

$\vec{y} = \begin{pmatrix} y_1 \\ \frac{10}{3} y_1 \end{pmatrix} \in S_1$

$\alpha \vec{x} + \beta \vec{y} = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha \frac{10}{3} x_1 + \beta \frac{10}{3} y_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \frac{10}{3} (\alpha x_1 + \beta y_1) \end{pmatrix} \in S_1$

$\Rightarrow S_1$ is a subspace of $\mathbb{R}^2$.

(b) $\vec{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \in S_2$ because $-2 + 2 \cdot 1^2 = 0$

$2\vec{x} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \notin S_2$

$\Rightarrow S_2$ is not a subspace of $\mathbb{R}^2$

8. Assume A is 6×3 matrix of rank 3. What is the dimension of $Col(A)$? What is the dimension of $Nul(A)$? What is the dimension of $Row(A)$? Explain your answers.

$$\dim(Col(A)) = \text{rank}(A) = 3$$
$$\dim(Null(A)) = 3 - 3 = 0$$
$$\dim(Row(A)) = \text{rank}(A) = 3$$

9(optional). Assume A and B are $m \times n$ matrices. Prove that $rank(A+B) \le rank(A) + rank(B)$.

Assume that ~~A~~ $A_1 = \{\vec{u}_1, \dots \vec{u}_r\}$ is a basis for $Col(A)$ and $B_1 = \{\vec{v}_1, \dots \vec{v}_s\}$ is a basis for $Col(B)$

In such a case $r = rank(A)$, $s = rank(B)$

Let's show that $Col(A+B) \subseteq span(A \cup B)$.

If $\vec{x} \in Col(A+B)$

$$\Rightarrow \vec{x} = \sum_{i=1}^{n} c_i(\underbrace{\vec{a}_i}_{\text{columns of } A} + \underbrace{\vec{b}_i}_{\text{columns of } B}) = \sum_{i=1}^{n} c_i \left(\sum_{j=1}^{r} d_{ij} \vec{u}_j + \sum_{k=1}^{s} \beta_{ik} \vec{v}_k \right),$$

i.e. $\vec{x}$ is a lin combination of vectors from A_1 and B_1.

$\Rightarrow A_1 \cup B_1$ spans $Col(A+B) \Rightarrow Col(A+B) \subseteq span(A \cup B)$

$\Rightarrow$ basis for $Col(A+B) \subseteq A \cup B$.

So, $rank(A+B)$ = the number of vector in a basis for $Col(A+B) \le$ the number of vectors$_1$ + the number of vector in $B_1 = r + s$

$= rank(A) + rank(B)$ $\blacksquare$.