HW 01 Solution.

math 32 1

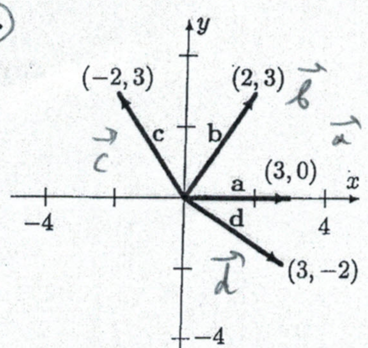
1

1.1

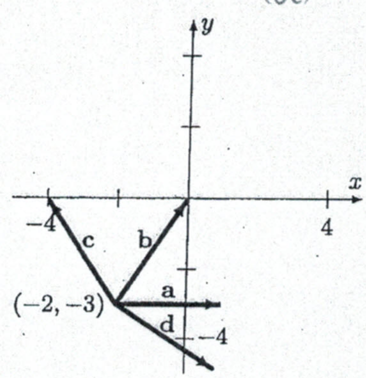
(a) $\vec{a} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ (d) $\vec{d} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(d)

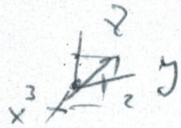
1.



2.



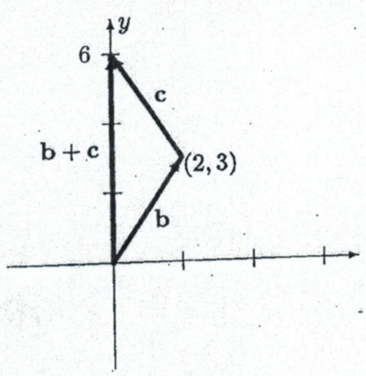
3 6 $\vec{b} = (3, 2, 1)$



4. (a) Following Example 1.1, we have the following:
If $[0, 2, 0]$ is translated to $\overrightarrow{BC}$ where $C = (4, 5, 6)$,
then we must have $B = (4 - 0, 5 - 2, 6 - 0) = (4, 3, 6)$.
Note: Unlike Example 1.1, we subtract $[0, 2, 0]$ instead of adding $[0, 2, 0]$. Why?

(b) Likewise, if $[3, 2, 1]$ is translated, $B = (4 - 3, 5 - 2, 6 - 1) = (1, 3, 5)$.

8. $\mathbf{b} + \mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$



10. $\mathbf{a} + \mathbf{d} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

②

12. $-2\mathbf{c} + 3\mathbf{b} + \mathbf{d} = -2\,[1, -2, 1] + 3\,[3, 2, 1] + [-1, -1, -2] = [+6, +9, -1]$.

16. $-3(\mathbf{a} - \mathbf{c}) + 2(\mathbf{a} + 2\mathbf{b}) + 3(\mathbf{c} - \mathbf{b}) \overset{\substack{\text{property e.}\\ \text{distributivity}}}{=} (-3\mathbf{a} + 3\mathbf{c}) + (2\mathbf{a} + 4\mathbf{b}) + (3\mathbf{c} - 3\mathbf{b}) \overset{\substack{\text{property b.}\\ \text{associativity}}}{=}$
$(-3\mathbf{a} + 2\mathbf{a}) + (4\mathbf{b} - 3\mathbf{b}) + (3\mathbf{c} + 3\mathbf{c}) = -\mathbf{a} + \mathbf{b} + 6\mathbf{c}$.

1.2

2. Following Example 1.8, $\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 3 \cdot 4 + (-2) \cdot 6 = 12 - 12 = 0.$

3. $\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1 = 2 + 6 + 3 = 11.$

5. $\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -\sqrt{2} \\ 0 \\ -5 \end{bmatrix} = 1 \cdot 4 + (\sqrt{2}) \cdot (-\sqrt{2}) + \sqrt{3} \cdot 0 + 0 \cdot (-5) = 4 - 2 = 2.$

7. In the remarks prior to Example 1.11, we note that finding a unit vector $\mathbf{v}$ in the same direction as a given vector $\mathbf{u}$ is called ***normalizing*** the vector $\mathbf{u}$. Therefore, we proceed as in Example 1.12:

$\|\mathbf{u}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$, so a unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$ is

$\mathbf{v} = (1/\|\mathbf{u}\|)\,\mathbf{u} = (1/\sqrt{5}) \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}.$

~~**10.** Following Example 1.12, we have:~~

~~$\|\mathbf{u}\| = \sqrt{(3.2)^2 + (-0.6)^2 + (-1.4)^2} = \sqrt{12.56}$, so unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$~~

~~is $\mathbf{v} = (1/\|\mathbf{u}\|)\,\mathbf{u} = (1/\sqrt{12.56}) \begin{bmatrix} 3.2 \\ -0.6 \\ -1.4 \end{bmatrix} = \begin{bmatrix} 3.2/\sqrt{12.56} \\ -0.6/\sqrt{12.56} \\ -1.4/\sqrt{12.56} \end{bmatrix} \approx \begin{bmatrix} 0.2548 \\ -0.0478 \\ -0.1115 \end{bmatrix}.$~~

11. $\|\mathbf{u}\| = \sqrt{1^2 + (\sqrt{2})^2 + (\sqrt{3})^2 + 0^2} = \sqrt{6}$, so a unit vector in the direction of $\mathbf{u}$ is

$$\mathbf{v} = (1/\|\mathbf{u}\|)\,\mathbf{u} = (1/\sqrt{6}) \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ \sqrt{2}/\sqrt{6} \\ \sqrt{3}/\sqrt{6} \\ 0/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{3} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{3}/3 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}.$$

3

14. Following Example 1.13, we compute: $\mathbf{u}-\mathbf{v}=\begin{bmatrix}3\\-2\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}=\begin{bmatrix}-1\\-8\end{bmatrix}$, so

$$d(\mathbf{u},\mathbf{v})=\|\mathbf{u}-\mathbf{v}\|=-\sqrt{(-1)^2+(-8)^2}=\sqrt{65}.$$

15. Following Example 1.13, we compute: $\mathbf{u}-\mathbf{v}=\begin{bmatrix}1\\2\\3\end{bmatrix}-\begin{bmatrix}2\\3\\1\end{bmatrix}=\begin{bmatrix}-1\\-1\\2\end{bmatrix}$, so

$$d(\mathbf{u},\mathbf{v})=\|\mathbf{u}-\mathbf{v}\|=-\sqrt{(-1)^2+(-1)^2+2^2}=\sqrt{6}.$$

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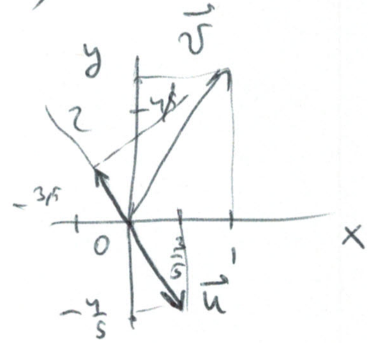
(b) $\mathbf{u}\cdot\mathbf{v}$ is a scalar, while $\mathbf{w}$ is a vector.
Thus, $\mathbf{u}\cdot\mathbf{v}+\mathbf{w}$ adds a scalar to a vector, which is not a defined operation.

24. As in Example 1.14, we begin by calculating $\mathbf{u}\cdot\mathbf{v}$ (if $\mathbf{u}\cdot\mathbf{v}=0$, we're done. Why?):
$\mathbf{u}\cdot\mathbf{v}=3\cdot(-1)+0\cdot 1=-3$, $\|\mathbf{u}\|=\sqrt{3^2+0^2}=\sqrt{9}=3$, $\|\mathbf{v}\|=\sqrt{(-1)^2+1^2}=\sqrt{2}$.
So, $\cos\theta=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=\dfrac{-3}{3\sqrt{2}}=-\dfrac{\sqrt{2}}{2}$ and $\theta=\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)=\dfrac{3\pi}{4}$ radians or 135°.

25. As in Example 1.14, we begin by calculating $\mathbf{u}\cdot\mathbf{v}$ (because if $\mathbf{u}\cdot\mathbf{v}=0$ we're done. Why?):
$\mathbf{u}\cdot\mathbf{v}=2\cdot 1+(-1)\cdot(-2)+1\cdot(-1)=2+2-1=3$,
$\|\mathbf{u}\|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{6}$, and $\|\mathbf{v}\|=\sqrt{1^2+(-2)^2+(-1)^2}=\sqrt{6}$.
Therefore, $\cos\theta=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=\dfrac{3}{\sqrt{6}\sqrt{6}}=\dfrac{1}{2}$, so $\theta=\cos^{-1}\left(\dfrac{1}{2}\right)=\dfrac{\pi}{3}$ radians or 60°.

30. To show $\triangle ABC$ is right, we need only show one pair of its sides meet at a right angle.
So, we let $\mathbf{u}=\overrightarrow{AB}$, $\mathbf{v}=\overrightarrow{BC}$, and $\mathbf{w}=\overrightarrow{AC}$, then by the definition of *orthogonal* given prior to Example 1.16, we need only show $\mathbf{u}\cdot\mathbf{v}$, or $\mathbf{u}\cdot\mathbf{w}$, or $\mathbf{v}\cdot\mathbf{w}=0$.
Following Example 1.1 of Section 1.1, we calculate the sides of $\triangle ABC$:
$\mathbf{u}=\overrightarrow{AB}=[1-(-3),0-2]=[4,-2]$, $\mathbf{v}=\overrightarrow{BC}=[4-1,6-0]=[3,6]$,
$\mathbf{w}=\overrightarrow{AC}=[4-(-3),6-2]=[7,4]$, so $\mathbf{u}\cdot\mathbf{v}=4\cdot 3+(-2)\cdot 6=12-12=0\Rightarrow$
The angle between $\mathbf{u}=\overrightarrow{AB}$ and $\mathbf{v}=\overrightarrow{BC}$ is 90° $\Rightarrow\triangle ABC$ is a right triangle.
Note: It is obvious that $\mathbf{v}$ is not orthogonal to $\mathbf{w}$. Why?

41 $\vec{u} = \begin{pmatrix} 3/5 \\ -\frac{4}{5} \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



$\vec{u}\cdot\vec{v} = \frac{3}{5} - \frac{8}{5} = -1$

$\vec{u}\cdot\vec{u} = \frac{9}{25} + \frac{16}{25} = 1$

$\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\cdot\vec{u} = \frac{-1}{1}\begin{pmatrix} 3/5 \\ -\frac{4}{5} \end{pmatrix} = \begin{pmatrix} -3/5 \\ 4/5 \end{pmatrix}$

43 $\vec{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 2 \\ -3 \\ -1 \\ -2 \end{pmatrix}$

$\vec{u}\cdot\vec{v} = 2 + 3 - 1 + 2 = 6$

$\vec{u}\cdot\vec{u} = 1 + 1 + 1 + 1 = 4$

$\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\cdot\vec{u} = \frac{6}{4}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -3/2 \\ 3/2 \\ -\frac{3}{2} \end{pmatrix}$

51 Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u}\cdot\vec{v}=0$

thus $\vec{u}\cdot\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}\cdot\begin{pmatrix} a \\ b \end{pmatrix} = ax+by=0$

Solve for y assuming $b\neq 0$: $y=-\frac{ax}{b}$

$\Rightarrow \vec{u} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -\frac{ax}{b} \end{pmatrix}$

If $b=0 \Rightarrow \cancel{x}=0$.

then take $x=kb \Rightarrow \vec{u} = \begin{pmatrix} kb \\ -ak \end{pmatrix} = k\begin{pmatrix} b \\ -a \end{pmatrix}$

which includes case $b=0$.

here k is arbitrary scalar.

55 We need to show $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|\mathbf{v}-\mathbf{u}\| = d(\mathbf{v},\mathbf{u})$.
If we let $c=-1$ in Theorem 1.3(b), then $\|-\mathbf{w}\| = \|\mathbf{w}\|$. We use this key fact below.

PROOF:	$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$	By definition
	$= \|-(\mathbf{v}-\mathbf{u})\|$	By the fact that $(x-y) = -(y-x)$
	$= \|\mathbf{v}-\mathbf{u}\|$	By $\|-\mathbf{w}\| = \|\mathbf{w}\|$ (key fact)
	$= d(\mathbf{v},\mathbf{u})$.	By definition

59 We need to show $\|\mathbf{u}-\mathbf{v}\| \geq \|\mathbf{u}\| - \|\mathbf{v}\|$. That is, $\|\mathbf{u}\| \leq \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}\|$.
This follows immediately from Theorem 1.5, $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\|+\|\mathbf{y}\|$, with $\mathbf{x}=\mathbf{u}-\mathbf{v}$ and $\mathbf{y}=\mathbf{v}$.

66 From Example 1.9 and the fact that $\mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2$, we have $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2$.
Taking the square root of both sides yields $\|\mathbf{u}+\mathbf{v}\| = \sqrt{\|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{u}\|^2}$.
Substituting in the given values of $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = \sqrt{3}$, and $\mathbf{u}\cdot\mathbf{v} = 1$
gives us $\|\mathbf{u}+\mathbf{v}\| = \sqrt{2^2 + 2(1) + (\sqrt{3})^2} = \sqrt{4+2+3} = \sqrt{9} = 3$.

7 (optional) (a) The Cauchy-Schwarz Inequality tells us $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$.
Squaring both sides, we get $|\mathbf{u}\cdot\mathbf{v}|^2 \le \|\mathbf{u}\|^2\|\mathbf{v}\|^2$.
In $\mathbb{R}^2$ with $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, this becomes $(u_1v_1 + u_2v_2)^2 \le (u_1^2 + u_2^2)(v_1^2 + v_2^2) \Leftrightarrow$
$0 \le (u_1^2 + u_2^2)(v_1^2 + v_2^2) - (u_1v_1 + u_2v_2)^2 \Leftrightarrow 0 \le u_1^2v_2^2 + u_2^2v_1^2 - 2u_1u_2v_1v_2 \Leftrightarrow$
$0 \le \frac{1}{2}(u_1v_2 - u_2v_1)^2 + \frac{1}{2}(u_2v_1 - u_1v_2)^2$.
Since the final statement is true, all the statements are true.

(b) Let $\mathbf{u}$ and $\mathbf{v}$ be elements of $\mathbb{R}^3$. Then $|\mathbf{u}\cdot\mathbf{v}|^2 \le \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \Leftrightarrow$
$(u_1v_1 + u_2v_2 + u_3v_3)^2 \le (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) \Leftrightarrow$
$0 \le (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \Leftrightarrow$
$0 \le u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 - 2u_1v_1u_2v_2 - 2u_1v_1u_3v_3 - 2u_2v_2u_3v_3 \Leftrightarrow$
$0 \le \frac{1}{2}(u_1v_2 - u_2v_1)^2 + \frac{1}{2}(u_2v_1 - u_1v_2)^2 + \frac{1}{2}(u_1v_3 - u_3v_1)^2$
$+ \frac{1}{2}(u_3v_1 - u_1v_3)^2 + \frac{1}{2}(u_2v_3 - u_3v_2)^2 + \frac{1}{2}(u_3v_2 - u_2v_3)^2$.
Since the final statement is true, all the statements are true.