(1.3)

1. Following Example 1.27 we will:
 (a) find the normal form by substituting into $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ and
 (b) find the general form by computing those dot products.

(a) $\mathbf{n} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow$ The normal form is $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$.

(b) $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 3x + 2y$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \Rightarrow$ The general form is $3x + 2y = 0$.

②. Following Example 1.27 we will:
 (a) find the normal form by substituting into $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ and
 (b) find the general form by computing those dot products.

(a) $\mathbf{n} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\Rightarrow$ Normal form is $\begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -5$

(b) $\begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 3x - 4y$ and $\begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -5 \Rightarrow$ The general form is $3x - 4y = -5$

④ Following Example 1.28 we will:
 (a) find the vector form by substituting into $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ and
 (b) find the parametric form by equating components.

(a) $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$, and $\mathbf{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\Rightarrow$ The vector form is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) The vector form in (a) implies the parametric form is $\begin{matrix} x = -4 + t \\ y = 4 + t \end{matrix}$.

⑦. Following Example 1.30 we will:
 (a) find the normal form by substituting into $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ and
 (b) find the general form by computing those dot products.

(a) $\mathbf{n} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\Rightarrow$ The normal form is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 2$.

(b) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3x + 2y + z$ and $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 2 \Rightarrow$ The general form is $3x + 2y + z = 2$.

9. Following Example 1.3, we will:
   (a) find the vector form by substituting into $x = p + su + tv$ and
   (b) find the parametric form by equating components.

   (a) $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $p = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, and $v = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow$

   The vector form is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$.

   (b) The vector form in (a) implies the parametric form is $\begin{aligned} x &= 2s - 3t \\ y &= s + 2t \\ z &= 2s + t \end{aligned}$.

11. Following Example 1.3, we realize we may choose any point on ℓ, $\quad p = (1, -2) \quad Q = (3, 0)$
    so we will use P (Q would also be fine).

    A convenient direction vector is $d = \overrightarrow{PQ} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (or any scalar multiple of this).

    Thus we obtain: $\quad x = p + td$
    $$= \begin{bmatrix} 1 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

14. Following Example 1.3, we realize we need to find two direction vectors, u and v.
    Since $P = (1, 1, 0)$, $Q = (1, 0, 1)$, and $R = (0, 1, 1)$ lie in plane $\mathscr{P}$, we compute:

    $$u = \overrightarrow{PQ} = q - p = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad v = \overrightarrow{PR} = r - p = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

    Since u and v are not scalar multiples of each other, they will serve as direction vectors.
    If u and v were scalar multiples of each other, we would not have a plane but simply a line.

    Therefore, we have the vector equation of $\mathscr{P}$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

15. The parametric equations and associated vector forms $x = p + td$ found below are *not* unique.

    (a) As in the remarks prior to Example 1.27 we begin by letting $x = t$.
        When we substitute $x = t$ into $y = 3x - 1$, we get $y = 3(t) - 1$. So, we have the following:

        Parametric equations $\begin{aligned} x &= t \\ y &= -1 + 3t \end{aligned}$ and vector form $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

18. Given **d** is the direction vector of line ℓ and **n** is the normal vector to the plane $\mathscr{P}$, we have:
If **d** and **n** are orthogonal which implies $\mathbf{d} \cdot \mathbf{n} = 0$, then line ℓ is parallel to plane $\mathscr{P}$.
If **d** and **n** are parallel which implies $\mathbf{d} = c\mathbf{n}$ (scalar multiples), then ℓ is perpendicular to $\mathscr{P}$.

(a) Since the general form of $\mathscr{P}$ is $2x + 3y - z = 1$, its normal vector is $\mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \mathbf{d}$.

Since $\mathbf{d} = 1\,\mathbf{n}$, ℓ is perpendicular to $\mathscr{P}$.

(b) Since the general form of $\mathscr{P}$ is $4x - y + 5z = 0$, its normal vector is $\mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$.

Since $\mathbf{d} \cdot \mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} = 2 \cdot 4 + 3 \cdot (-1) + (-1) \cdot 5 = 0$, ℓ is parallel to $\mathscr{P}$.

20. Since the vector form is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we use the given information to determine **p** and **d**.

The general equation of the given line is $2x - 3y = 1$, so its normal vector is $\mathbf{n} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Our line is perpendicular to the given line, so it has direction vector $\mathbf{d} = \mathbf{n} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Furthermore, since our line passes through the point $P = (2, -1)$, we have $\mathbf{p} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

So, the vector form of the line perpendicular to $2x - 3y = 1$ through the point $P = (2, -1)$ is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

23. Since the vector form is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we use the given information to determine **p** and **d**.

A line with parametric equations $\begin{array}{l} x = a + et \\ y = b + ft \\ z = c + gt \end{array}$ has vector form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + t \begin{bmatrix} e \\ f \\ g \end{bmatrix}$.

Therefore, its direction vector is $\mathbf{d} = \begin{bmatrix} e \\ f \\ g \end{bmatrix}$. We use this key observation below.

Since the given line has parametric equations $\begin{array}{l} x = 1 - t \\ y = 2 + 3t \\ z = -2 - t \end{array}$,

it has vector form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$. So, its direction vector is $\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$.

Since our line is parallel to the given line, its direction vector is also $\mathbf{d} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$.

Furthermore, since our line passes through the point $P = (-1, 0, 3)$, we have $\mathbf{p} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$.

So, the vector form of the line parallel to the given line through $P = (-1, 0, 3)$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}.$$

(optional)

25. Following Example 1.30 we will determine the general equations in two simple steps:
First, we will use Figure 1.31 in Section 1.2 to find a normal vector **n** and a point vector **p**.
Then we will substitute into $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ and compute the dot products to find the equations.

(a) We start with $\mathscr{P}_1$ determined by the face of the cube in the yz-plane.

It is clear that a normal vector for $\mathscr{P}_1$ is $\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or any vector parallel to the x-axis.

Also we see that $\mathscr{P}_1$ passes through the origin $P = (0,0,0)$, so we set $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Substituting into $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ yields $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or $1 \cdot x + 0 \cdot y + 0 \cdot z = 0$.

So, the general equation for $\mathscr{P}_1$ determined by the face in the yz-plane is $x = 0$.
Likewise, the general equation for $\mathscr{P}_2$ determined by the face in the xz-plane is $y = 0$
and the general equation for $\mathscr{P}_3$ determined by the face in the xy-plane is $z = 0$.
We have found equations for the planes that pass through the origin.
We will use this information to find equations for the planes that pass through $(1,1,1)$.
We begin with $\mathscr{P}_4$ passing through the face parallel to the face in the yz-plane.

Since $\mathscr{P}_4$ is parallel to the face in the yz-plane, its normal vector is $\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

As previously noted $\mathscr{P}_4$ passes through the point $P = (1,1,1)$, so we set $\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Substituting into $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ yields $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or $1 \cdot x + 0 \cdot y + 0 \cdot z = 1$.

So, the general equation for $\mathscr{P}_4$ is $x = 1$.
Likewise, the general equations for $\mathscr{P}_5$ and $\mathscr{P}_6$ are $y = 1$ and $z = 1$ respectively.

(b) We will use the given information to determine $\mathbf{n}$ and $\mathbf{p}$, then compute $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$.

We begin by observing the two key facts that will enable us to find $\mathbf{n}$ and $\mathbf{p}$:

Two planes $\mathscr{P}_1$, $\mathscr{P}$ are perpendicular if their normal vectors are orthogonal, so $\mathbf{n}_1 \cdot \mathbf{n} = 0$.

Every vector $\mathbf{u}$ in the plane $\mathscr{P}_1$ is orthogonal to its normal vector $\mathbf{n}_1$, so $\mathbf{n}_1 \cdot \mathbf{u} = 0$.

Condition 1: Our plane must be perpendicular to the xy-plane, so $\mathbf{n}_1 \cdot \mathbf{n} = 0$. From (a),

$$\mathbf{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ so } \mathbf{n}_1 \cdot \mathbf{n} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \Rightarrow z = 0. \text{ So, } \mathbf{n}_1 \text{ is of the form } \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

Condition 2: $\mathbf{n}_1$ must be perpendicular to the vector $\mathbf{u}$ from the origin to $(1,1,1)$.

Since $\mathbf{u} = \begin{bmatrix} 1-0 \\ 1-0 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, we have $\mathbf{n}_1 \cdot \mathbf{n} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \Rightarrow x + y = 0 \Rightarrow y = -x.$

So, $\mathscr{P}_1$ must be of the form $\mathbf{n}_1 = \begin{bmatrix} x \\ -x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Letting $x = 1$ yields $\mathbf{n}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

As previously noted $\mathscr{P}_1$ passes through the origin $P = (0,0,0)$, so we set $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Now $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ yields $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or $1 \cdot x + (-1) \cdot y + 0 \cdot z = 0.$

Therefore, the general equation for the plane perpendicular to the xy-plane and containing the diagonal from the origin to $(1,1,1)$ is $x - y = 0$.

(c) As above, use $\mathbf{u} = [0,1,1]$ and $\mathbf{v} = [1,0,1]$ from Example 1.15 of Section 1.2 to find $\mathbf{n}$.

From $\mathbf{n} \cdot \mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 \Rightarrow y + z = 0 \Rightarrow y = -z.$

From $\mathbf{n} \cdot \mathbf{v} = \begin{bmatrix} x \\ -z \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0 \Rightarrow x + z = 0 \Rightarrow x = -z.$

So, the normal vector $\mathbf{n} = \begin{bmatrix} -z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. When $z = -1$, we have $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

It is obvious the side diagonals pass through the origin $P = (0,0,0)$, so we set $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Now $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ yields $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or $1 \cdot x + 1 \cdot y + (-1) \cdot z = 0.$

The general equation for the plane containing the side diagonals is $x + y - z = 0$.

(28) We will follow Example 1.32 then use $d(Q, \ell) = \dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$ and compare results.

Even though the formula $d(Q, \ell) = \dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$ was developed for planes,

it can work for lines in $\mathbb{R}^3$ with the proper choice of $\mathbf{n} = [a, b, c]$.

Comparing $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$ to $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we see $P = (1, 1, 1)$ and $\mathbf{d} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$.

As suggested by Figure 1.66 we need to calculate the length of $\overrightarrow{RQ}$,
where R is the point on ℓ at the foot of the perpendicular from Q.

Now if we let $\mathbf{v} = \overrightarrow{PQ}$, then $\overrightarrow{PR} = \text{proj}_{\mathbf{d}}(\mathbf{v})$ and $\overrightarrow{RQ} = \mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})$.

Step 1. $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$.

Step 2. $\text{proj}_{\mathbf{d}}(\mathbf{v}) = \left(\dfrac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}} \right) \mathbf{d} = \left(\dfrac{(-2) \cdot (-1) + (3) \cdot (-1)}{(-2) \cdot (-2) + 3 \cdot 3} \right) \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/13 \\ 0 \\ -3/13 \end{bmatrix}$.

Step 3. The vector we want is $\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v}) = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 2/13 \\ 0 \\ -3/13 \end{bmatrix} = \begin{bmatrix} -15/13 \\ 0 \\ -10/13 \end{bmatrix}$.

Step 4. The distance $d(Q, \ell)$ from Q to ℓ is $\| \mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v}) \| = \left\| \begin{bmatrix} -15/13 \\ 0 \\ -10/13 \end{bmatrix} \right\|$.

So Theorem 1.3(b) implies $\| \mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v}) \| = \dfrac{5}{13} \left\| \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \right\| = \dfrac{5}{13} \sqrt{9 + 4} = \dfrac{5\sqrt{13}}{13}$.

Now in order to calculate $d(Q, \ell) = \dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$ we need to put ℓ into general form.

The appropriate choice of $\mathbf{n}$ mentioned at the top follows from the following observation:

Vector $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$ found using Theorem 1.3(b) in Step 4 is orthogonal to $\mathbf{d}$, so let $\mathbf{n} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$.

From $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$, $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ so $3x + 2z = 5$ and $a = 3$, $b = 0$, $c = 2$, $d = 5$.

Furthermore, since $Q = (0, 1, 0) = (x_0, y_0, z_0)$ we have $x_0 = 0$, $y_0 = 1$, and $z_0 = 0$.

So $d(Q, \ell) = \dfrac{|0 + 0 + 0 - 5|}{\sqrt{3^2 + 2^2}} = \dfrac{5}{\sqrt{13}} = \dfrac{5\sqrt{13}}{13}$ exactly as we found by following Example 1.32

(29.) We will follow Example 1.33 then use $d(Q, \mathcal{P}) = \dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$ and compare results.

By definition $ax + by + cz = d$ implies $\mathbf{n} = [a, b, c]$, so $x + y - z = 0$ implies $\mathbf{n} = [1, 1, -1]$.

As suggested by Figure 1.64, we need to calculate the length of $\overrightarrow{RQ} = \text{proj}_{\mathbf{n}}(\mathbf{v})$, where $\mathbf{v} = \overrightarrow{PQ}$.

Step 1. By trial and error, we find $P = (1, 0, 1)$ satisfies $x + y - z = 0$.

Step 2. $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Step 3. $\text{proj}_{\mathbf{n}}(\mathbf{v}) = \left(\dfrac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{n}} \right) \mathbf{n} = \left(\dfrac{1 \cdot 1 + 1 \cdot 2 - 1 \cdot 1}{1^2 + 1^2 + (-1)^2} \right) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \dfrac{2}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}$.

Step 4. The distance from Q to $\mathcal{P}$ is $\|\text{proj}_{\mathbf{n}}(\mathbf{v})\| = \left\| \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix} \right\| = \dfrac{2}{3} \left\| \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\| = \dfrac{2\sqrt{3}}{3}$.

Now for $d(Q, \mathcal{P}) = \dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$ we need identify a, b, c, d, and x_0, y_0, z_0.

Since $x + y - z = 0$, $a = 1$, $b = 1$, $c = -1$, $d = 0$. From $Q = (2, 2, 2)$, $x_0 = y_0 = z_0 = 2$.

So $d(Q, \mathcal{P}) = \dfrac{|2 + 2 - 2 + 0|}{\sqrt{1^2 + 1^2 + (-1)^2}} = \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$ as we found by following Example 1.33.

(31.) Similar to Example 1.33, Figure 1.66 suggests we let $\mathbf{v} = \overrightarrow{PQ}$, then $\mathbf{w} = \overrightarrow{PR} = \text{proj}_{\mathbf{d}}(\mathbf{v})$.

Comparing $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we see ℓ has $P = (-1, 2)$ and $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Step 1. $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Step 2. $\mathbf{w} = \text{proj}_{\mathbf{d}}(\mathbf{v}) = \left(\dfrac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}} \right) \mathbf{d} = \left(\dfrac{1 \cdot 3 + (-1) \cdot 0}{1 \cdot 1 + (-1) \cdot (-1)} \right) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \dfrac{3}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix}$.

Step 3. So, $\mathbf{r} = \mathbf{p} + \overrightarrow{PR} = \mathbf{p} + \text{proj}_{\mathbf{d}}(\mathbf{v}) = \mathbf{p} + \mathbf{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

Therefore, the point R on ℓ that is closest to Q is $\left(\frac{1}{2}, \frac{1}{2} \right)$.

35. Since the given lines ℓ_1 and ℓ_2 are parallel, we can simply choose Q on ℓ_1, P on ℓ_2. Following Example 1.32, we have:

From ℓ_1, $Q = (1,1)$. From ℓ_2, we have $P = (5,4)$, $\mathbf{d} = [-2,3]$, and $\mathbf{n} = [3,2] = [a,b]$.

Step 1. $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = [1,1] - [5,4] = [-4,-3]$.

Step 2. $\text{proj}_\mathbf{d}(\mathbf{v}) = \left(\dfrac{\mathbf{d}\cdot\mathbf{v}}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d} = \left(\dfrac{(-2)\cdot(-4)+3\cdot(-3)}{(-2)^2+3^2}\right)\begin{bmatrix}-2\\3\end{bmatrix} = -\dfrac{1}{13}\begin{bmatrix}-2\\3\end{bmatrix} = \begin{bmatrix}2/13\\-3/13\end{bmatrix}$.

Step 3. The vector we want is $\mathbf{v} - \text{proj}_\mathbf{d}(\mathbf{v}) = \begin{bmatrix}-4\\-3\end{bmatrix} - \begin{bmatrix}2/13\\-3/13\end{bmatrix} = \begin{bmatrix}-54/13\\-36/13\end{bmatrix}$.

Step 4. The distance $d(Q,\ell_2)$ from ℓ_1 to ℓ_2 is $\|\mathbf{v} - \text{proj}_\mathbf{d}(\mathbf{v})\| = \left\|\begin{bmatrix}-54/13\\-36/13\end{bmatrix}\right\|$.

So Theorem 1.3(b) implies $\|\mathbf{v} - \text{proj}_\mathbf{d}(\mathbf{v})\| = \dfrac{18}{13}\left\|\begin{bmatrix}3\\2\end{bmatrix}\right\| = \dfrac{18}{13}\sqrt{4+9} = \dfrac{18\sqrt{13}}{13}$.

From $\mathbf{n}\cdot\mathbf{p} = \begin{bmatrix}3\\2\end{bmatrix}\cdot\begin{bmatrix}5\\4\end{bmatrix} = 23$, $c = 23$. Since $Q = (1,1) = (x_0,y_0)$, we have $x_0 = y_0 = 1$.

Now compare: $d(\ell_1,\ell_2) = d(Q,\ell_2) = \dfrac{|ax_0+by_0-c|}{\sqrt{a^2+b^2}} = \dfrac{|3+2-23|}{\sqrt{3^2+2^2}} = \dfrac{18}{\sqrt{13}} = \dfrac{18\sqrt{13}}{13}$.

37. Since the given planes $\mathscr{P}_1$ and $\mathscr{P}_2$ are parallel, we can simply choose Q in $\mathscr{P}_1$, P in $\mathscr{P}_2$. Following Example 1.33 we have:

Step 1. Since $2x + y - 2z = 0$, $Q = (0,0,0)$ is on $\mathscr{P}_1$.

Since $2x + y - 2z = 5$, $P = (0,5,0)$ is on $\mathscr{P}_2$ and $\mathbf{n} = [2,1,-2] = [a,b,c]$.

Step 2. $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = [0,0,0] - [0,5,0] = [0,-5,0]$

Step 3. $\text{proj}_\mathbf{n}(\mathbf{v}) = \left(\dfrac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \left(\dfrac{2\cdot0-1\cdot5-2\cdot0}{2^2+1^2+(-2)^2}\right)\begin{bmatrix}2\\1\\-2\end{bmatrix} = -\dfrac{5}{9}\begin{bmatrix}2\\1\\-2\end{bmatrix} = \begin{bmatrix}-10/9\\-5/9\\10/9\end{bmatrix}$.

Step 4. The distance $d(Q,\mathscr{P}_2)$ from $\mathscr{P}_1$ to $\mathscr{P}_2$ is $\|\mathbf{v} - \text{proj}_\mathbf{n}(\mathbf{v})\| = \left\|\begin{bmatrix}-10/9\\-5/9\\10/9\end{bmatrix}\right\|$.

So Theorem 1.3(b) implies $\|\text{proj}_\mathbf{n}(\mathbf{v})\| = \dfrac{5}{9}\left\|\begin{bmatrix}2\\1\\-2\end{bmatrix}\right\| = \dfrac{5}{9}\sqrt{4+1+4} = \dfrac{5}{3}$.

From $\mathbf{n}\cdot\mathbf{p} = d = [2,1,-2]\cdot[0,5,0] = 5$. Since $Q = (0,0,0)$, $x_0 = y_0 = z_0 = 0$.

Now compare: $d(Q,\mathscr{P}_2) = \dfrac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}} = \dfrac{|0+0+0-5|}{\sqrt{2^2+1^2+(-2)^2}} = \dfrac{5}{\sqrt{9}} = \dfrac{5}{3}$.

41. *(optional)* We will apply the formula from Exercise 39, $d(B, \ell) = \dfrac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|}$.

Step 1. We select $B = (x_0, y_0)$ on ℓ_1 so that $\mathbf{n} \cdot \mathbf{b} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = ax_0 + by_0 = c_1$.

Step 2. We select A on ℓ_2 so that $\mathbf{n} \cdot \mathbf{a} = c_2$

Step 3. Set $\mathbf{v} = \mathbf{b} - \mathbf{a}$, then $d(B, \mathscr{P}) = \dfrac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|} = \dfrac{|\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})|}{\|\mathbf{n}\|} = \dfrac{|\mathbf{n} \cdot \mathbf{b} - \mathbf{n} \cdot \mathbf{a}|}{\|\mathbf{n}\|} = \dfrac{|c_1 - c_2|}{\|\mathbf{n}\|}$.

42. *(optional)* We will apply the formula from Exercise 40, $d(B, \mathscr{P}) = \dfrac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|}$.

Step 1. We select $B = (x_0, y_0, z_0)$ on $\mathscr{P}_1$ so that $\mathbf{n} \cdot \mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = ax_0 + by_0 + cz_0 = d_1$.

Step 2. We select A on $\mathscr{P}_2$ so that $\mathbf{n} \cdot \mathbf{a} = d_2$.

Step 3. Set $\mathbf{v} = \mathbf{b} - \mathbf{a}$, then $d(B, \mathscr{P}) = \dfrac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|} = \dfrac{|\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})|}{\|\mathbf{n}\|} = \dfrac{|\mathbf{n} \cdot \mathbf{b} - \mathbf{n} \cdot \mathbf{a}|}{\|\mathbf{n}\|} = \dfrac{|d_1 - d_2|}{\|\mathbf{n}\|}$.

43. *(optional)* As in Example 1.21 of Section 1.2, we note that $\cos \theta = \dfrac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|}$.

So, given two planes $\mathscr{P}_1$ with $\mathbf{n}_1$ and $\mathscr{P}_2$ with $\mathbf{n}_2$, we have $\cos \theta = \dfrac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|}$.

Step 1. Since $\mathscr{P}_1$ has equation $x + y + z = 0$, $\mathbf{n}_1 = [1, 1, 1]$.

Since $\mathscr{P}_2$ has equation $2x + y - 2z = 0$, $\mathbf{n}_2 = [2, 1, -2]$.

Step 2. Therefore, $\mathbf{n}_1 = [1, 1, 1] \cdot [2, 1, -2] = 1 \cdot 2 + 1 \cdot 1 - 1 \cdot 2 = 1$,

$\|\mathbf{n}_1\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$, and $\|\mathbf{n}_2\| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$.

Step 3. So $\cos \theta = \dfrac{1}{3\sqrt{3}}$ and $\theta = \cos^{-1}\left(\dfrac{1}{3\sqrt{3}}\right) \approx 78.9°$.

(2.1)

1. We follow Example 2.1 and justify our assertion by applying the definition of *linear*.
$x - \pi y + \left(\sqrt[3]{5}\right) z = 0$ *is* linear *because* power of z is 1 and π, $\sqrt[3]{5}$ are constants.

2. We follow Example 2.1 and justify our assertion by applying the definition of *linear*.
$x^2 + y^2 + z^2 = 1$ is *not* linear *because* x, y, z occur to the power 2.

3. $x^{-1} + 7y + z = \sin \frac{\pi}{9}$ is *not* linear *because* x occurs to the power -1.

11. As in Example 2.2(a), we set $x = t$ and solve for y.
Setting $x = t$ in $3x - 6y = 0$ gives us $3t - 6y = 0$. Solving for y yields $6y = 3t \Rightarrow y = \frac{1}{2}t$.
So, we see the complete set of solutions can be written in the parametric form $[t, \frac{1}{2}t]$.

Note: We could have set $y = t$ to get $3x - 6t = 0$ and solved for x so $x = 2t$ and $[2t, t]$.

15. The lines intersect at $(3, -3)$,
so the unique solution is $[3, -3]$.
To solve, subtract 2nd from 1st $\Rightarrow$
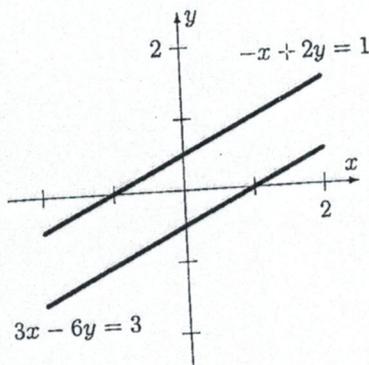$-x = -3 \Leftrightarrow x = 3$,
so substitution $\Rightarrow y = -3$.

$2x + y = 1$

$(3, -3)$

$x + y = 0$
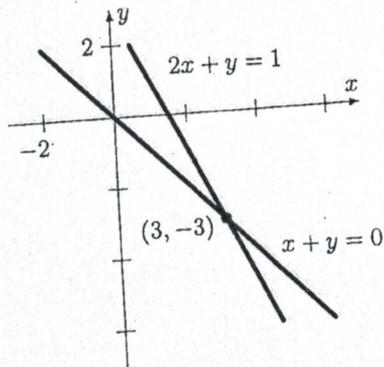
16. The lines intersect at $(3, -2)$,
so the unique solution is $[3, -2]$.
To solve, subtract 3×1st from 2nd $\Rightarrow$
$7y = -14 \Leftrightarrow y = -2$,
so substitution $\Rightarrow x = 3$.

$3x + y = 7$

$(3, -2)$

$x - 2y = 7$
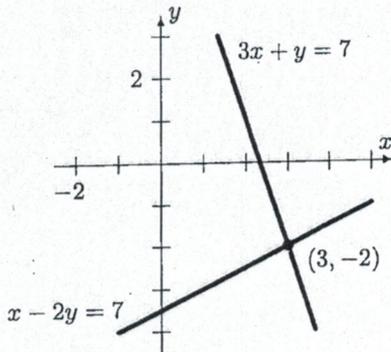
17. The lines are parallel $\Rightarrow$ no solution.
This system is inconsistent.
Add 3×2nd to 1st $\Rightarrow 0 = 6$.

$-x + 2y = 1$

$3x - 6y = 3$

22. We find the solution $[x_1, x_2, x_3] = [0, 0, 0]$ using back substitution.
Note: This follows immediately from the fact that all three equations are equal to zero.

28. As in the solution to Example 2.6, we create the augmented matrix from the coefficients.

The system $\begin{aligned} 2x_1 + 3x_2 - x_3 &= 1 \\ x_1 \quad\quad + x_3 &= 0 \\ -x_1 + 2x_2 - 2x_3 &= 0 \end{aligned}$ has $\begin{bmatrix} 2 & 3 & -1 & | & 1 \\ 1 & 0 & 1 & | & 0 \\ -1 & 2 & -2 & | & 0 \end{bmatrix}$ as its augmented matrix.