math 321 Hw 03 Solutions

2.2

1. No, this matrix is not in row echelon form. Why not? Give at least one reason.
The leading entry in row 3 appears to the left of the leading entry in row 2.

2. This matrix is in row echelon form, but not reduced row echelon form. Why not?.
There are many reasons. For example, the leading entry in row 1 is 7 not 1.

3. This matrix is in row echelon form, and also reduced row echelon form. Why is the 3 okay?
The 3 occurs in a column that does not contain a leading 1.

4. This matrix is in row echelon form, and also reduced row echelon form. Why are the 0s okay?
All three rows are zero, so no leading 1s are required.

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9. (a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. (b) ... $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 - R_3]{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

10. (a) $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \xrightarrow{R_2 - \frac{1}{2}R_1} \begin{bmatrix} 4 & 3 \\ 0 & -\frac{1}{2} \end{bmatrix}$ ... (b) ... $\begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{4}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

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14. (a) $\begin{bmatrix} -2 & -4 & 7 \\ -3 & -6 & 10 \\ 1 & 2 & -3 \end{bmatrix} \xrightarrow[2R_3]{-2R_2} \begin{bmatrix} -2 & -4 & 7 \\ 6 & 12 & -20 \\ 2 & 4 & -6 \end{bmatrix} \xrightarrow[R_3 + R_1]{R_2 + 3R_1} \begin{bmatrix} -2 & -4 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} -2 & -4 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) Continuing from (a): $\begin{bmatrix} -2 & -4 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 7R_2} \begin{bmatrix} -2 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

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17. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1} \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_1 + \frac{7}{2}R_2} \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} = B.$

So A and B are row equivalent. Convert A into B by $R_2 - 2R_1$, $-\frac{1}{2}R_1$, $R_1 + \frac{7}{2}R_2$.

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19. Performing $R_2 + R_1$ and $R_1 + R_2$ does *not* leave rows 1 and 2 identical.
After performing $R_2 + R_1$ the second row is now $R_2' = R_2 + R_1$.
So $R_1 + R_2$ is actually $R_1 + R_2' = R_1 + (R_2 + R_1) = 2R_1 + R_2$.
Performing $R_2 + R_1$ and $R_1 + R_2$ simultaneously annuls their linearity.

25. We have the following system of equations: $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 4 \end{bmatrix}$.

We form the augmented matrix and row reduce it as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{array}\right] \xrightarrow[R_3-2R_2]{R_1+3R_3} \left[\begin{array}{ccc|c} 13 & -1 & 0 & 21 \\ 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & 4 \end{array}\right] \xrightarrow[-R_3]{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 13 & -1 & 0 & 21 \\ 0 & -1 & 1 & -4 \end{array}\right]$$

$$\xrightarrow[-R_2]{R_1-R_3} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ -13 & 1 & 0 & -21 \\ 0 & -1 & 1 & 4 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ -13 & 1 & 0 & -21 \\ 0 & -1 & 1 & -4 \end{array}\right] \xrightarrow{R_2+13R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & -1 & 1 & -4 \end{array}\right]$$

$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right].$$ So, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$.

26. We form the augmented matrix and row reduce it as follows:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 3 & 1 & 5 \\ 3 & 1 & 7 & 2 \end{array}\right] \xrightarrow{R_3-5R_1-2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 3 & 1 & 5 \\ 0 & 0 & 0 & -8 \end{array}\right]$$

The third row is equivalent to the equation $0 = -8$ which clearly has no solution. Therefore, the system is inconsistent.

Does $R_3 = 5R_1 + 2R_2$ (excluding constants) cause the system to be inconsistent?

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34. When there are 4 equations and 4 variables, if the solution exists it is unique. Why? Because the Rank Theorem (2.2) tells us there are $4 - 4 = 0$ free variables.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 4 & 10 \\ 1 & 3 & 6 & 10 & 20 \\ 1 & 4 & 10 & 20 & 35 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \text{The solution is } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

35. Begin by thinking of this system as $[A|\mathbf{x}]$, then determine rank A by inspection. Mentally performing $R_1 \leftrightarrow R_3$ to put matrix A into row echelon form, makes it obvious that rank $A = 3$ (because A has 3 nonzero rows).

Since rank $A = 3$, this is a system of 3 equations and 3 variables. Therefore, the system has a unique solution because there are $3 - 3 = 0$ free variables.

36. Begin by thinking of this system as $[A|\mathbf{x}]$, then determine rank A by inspection. Mentally performing $R_3 - 2R_2$ implies the equation $0 = 2$. This equation makes it obvious that this system has no solution.

Note: $R_3 = 2R_2$ implies rank $A = 2$. How does that relate to our answer?

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40. First row reduce the system $[A|\mathbf{x}]$ and then answers parts (a), (b), and (c).

$$\left[\begin{array}{cc|c} k & 2 & 3 \\ 2 & -4 & -6 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 2 & -4 & -6 \\ k & 2 & 3 \end{array}\right] \xrightarrow{R_2 - \frac{1}{2}kR_1} \left[\begin{array}{cc|c} 2 & -4 & -6 \\ 0 & 2+2k & 3+3k \end{array}\right]$$

(a) There are no values of k for which this system has no solution. Why?
The system has no solution when A has a zero row with corresponding constant $\neq 0$.
$2+2k=0 \Rightarrow k=-1$ is the only value of k that creates a row of all zeros.
But $k=-1 \Rightarrow$ the constant $3+3k=3-3=0$. What does this imply?

(b) The system has a unique solution for $k \neq -1$. Why?
From (a), we see when $k \neq -1$ then rank $A=2$. So, there are $2-2=0$ free variables

(c) The only value of k for which this system has infinitely many solutions is $k=-1$.
The system has infinitely many solutions when A has a zero row with constant $=0$.
From (a), we see this is exactly the case when $k=-1$.

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45. As in Example 2.14, find the line of intersection of $3x+2y+z=-1$, $2x-y+4z=5$.
First, observe that there will be a line of intersection. Why?
The normal vectors of the two planes, $[3,2,1]$ and $[2,-1,4]$ are not parallel.

The points that lie in the intersection of the two planes correspond to the points

in the solution the system:
$$\begin{array}{rcrcrcr} 3x & + & 2y & + & z & = & -1 \\ 2x & - & y & + & 4z & = & 5 \end{array}$$

Gauss-Jordan elimination yields:
$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & -1 \\ 2 & -1 & 4 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{9}{7} & \frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{17}{7} \end{array}\right]$$

Replacing variables, we have:
$$\begin{array}{rcl} x + \frac{9}{7}z & = & \frac{9}{7} \\ y - \frac{10}{7}z & = & -\frac{17}{7} \end{array} \Rightarrow \begin{array}{rcl} z & = & 1 - \frac{7}{9}x \\ y & = & -\frac{17}{7} + \frac{10}{7}z \end{array}$$

To eliminate fractions we set $x=9t$, so $z=1-\frac{7}{9}(9t)=1-7t$.
Substituting $z=1-7t$ into $y=-\frac{17}{7}+\frac{10}{7}z$ yields: $y=-\frac{17}{7}+\frac{10}{7}(1-7t)=-1-10t$.
Summarizing, we now have $x=9t$, $y=-1-10t$, and $z=1-7t$.

Therefore, the line is
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -10 \\ -9 \end{bmatrix}.$$

48. As in Example 2.15, if these lines intersect we need to determine the point of intersection.
As pointed out in that example, we need to change the parameter for the first line to s.
We want to find an $\mathbf{x}=[x,y,z]$ that satisfies both equations simultaneously.
That is, we want $\mathbf{x}=\mathbf{p}+s\mathbf{u}=\mathbf{q}+t\mathbf{v}$ or $s\mathbf{u}-t\mathbf{v}=\mathbf{q}-\mathbf{p}$.
Substituting the given $\mathbf{p}$, $\mathbf{q}$, $\mathbf{u}$, and $\mathbf{v}$ into $s\mathbf{u}-t\mathbf{v}=\mathbf{q}-\mathbf{p}$, we obtain the equations:

$$s\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{rcrcr} s & + & t & = & 3 \\ 2s & - & t & = & 0 \\ -s & & & = & -1 \end{array}$$

From this, the solution is easily found to be $s=1, t=2$.

Therefore, the point of intersection is:
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}.$$

Check that substituting $t=2$ into the other equation gives the same point.

2.3

1. As in Example 2.18, we want to find scalars x and y such that:

$x\begin{bmatrix}1\\-1\end{bmatrix}+y\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$ Expanding, we obtain the system: $\begin{array}{rcrcl} x &+& 2y &=& 1\\ -x &-& y &=& 2\end{array}$

We then row reduce the associated augmented matrix: $\left[\begin{array}{rr|r}1&2&1\\-1&-1&2\end{array}\right]\longrightarrow\left[\begin{array}{rr|r}1&0&-5\\0&1&3\end{array}\right]$

So the solution is $x=-5$, $y=2$, and the linear combination is $-5\begin{bmatrix}1\\-1\end{bmatrix}+3\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$.

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2. As in Example 2.18, we want to find scalars x and y such that:

$x\begin{bmatrix}4\\-2\end{bmatrix}+y\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$ Expanding, we obtain the system: $\begin{array}{rcrcl} 4x &-& 2y &=& 2\\ -2x &+& y &=& 1\end{array}$

We then row reduce the associated augmented matrix: $\left[\begin{array}{rr|r}4&-2&2\\-2&1&1\end{array}\right]\longrightarrow\left[\begin{array}{rr|r}4&-2&2\\0&0&2\end{array}\right]$

Since $0\neq 2$, this system clearly has no solution. So, what do we conclude?
We conclude that $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

We could have noted $\begin{bmatrix}4\\-2\end{bmatrix}=-2\begin{bmatrix}-2\\1\end{bmatrix}$, while $\begin{bmatrix}2\\1\end{bmatrix}$ is not a multiple of $\begin{bmatrix}-2\\1\end{bmatrix}$.

3. As in Example 2.18, we want to find scalars x and y such that:

$x\begin{bmatrix}1\\1\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$ Expanding, we obtain the system: $\begin{array}{rcrcl} x & & &=& 1\\ x &+& y &=& 2\\ & & y &=& 3\end{array}$

Since $x=1$ and $y=3$ implies $x+y\neq 2$, this system clearly has no solution.
Therefore, $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

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7. Applying Theorem 2.4, we check to see if $[A|\mathbf{b}]$ is consistent. Why?
Theorem 2.4 says $[A|\mathbf{b}]$ is consistent $\Leftrightarrow$ $\mathbf{b}$ is a linear combination of the columns of A.
That is exactly what is required for $\mathbf{b}$ to be in the span of the columns of A.

So we row reduce $[A|\mathbf{b}]=\left[\begin{array}{rr|r}1&2&5\\3&4&6\end{array}\right]\longrightarrow\left[\begin{array}{rr|r}1&0&-4\\0&1&\frac{9}{2}\end{array}\right]$ to see that it is consistent.

What do we conclude? The vector $\mathbf{b}$ is in the span of the columns of A.

In particular, the solution tells us the linear combination is $-4\begin{bmatrix}1\\3\end{bmatrix}+\frac{9}{2}\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}5\\6\end{bmatrix}$.
