2.3

9. As in Example 2.19, we must show $x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ can always be solved.

The augmented matrix is $\begin{bmatrix} 1 & 1 & | & a \\ 1 & -1 & | & b \end{bmatrix}$, and row reduction produces:

$$\begin{bmatrix} 1 & 1 & | & a \\ 1 & -1 & | & b \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 2 & 0 & | & a+b \\ 1 & -1 & | & b \end{bmatrix} \xrightarrow[-R_2]{1/2 R_1} \begin{bmatrix} 1 & 0 & | & (a+b)/2 \\ 0 & 1 & | & -b \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 & | & (a+b)/2 \\ -1 & 1 & | & (a-b)/2 \end{bmatrix}$$

We see that $x = (a+b)/2$ and $y = (a-b)/2$, so for any choice of a and b we have

$$\left(\frac{a+b}{2}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(\frac{a-b}{2}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{Check this!}$$

---

10. As in Example 2.19, we must show $x\begin{bmatrix} 3 \\ -2 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ can always be solved.

The augmented matrix is $\begin{bmatrix} 3 & 0 & | & a \\ -2 & 1 & | & b \end{bmatrix}$, and row reduction produces:

$$\begin{bmatrix} 3 & 0 & | & a \\ -2 & 1 & | & b \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 0 & | & a/3 \\ -2 & 1 & | & b \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & 0 & | & a/3 \\ 0 & 1 & | & (2a+3b)/3 \end{bmatrix}$$

We see that $x = a/3$ and $y = (2a+3b)/3$, so for any choice of a and b we have

$$\left(\frac{a}{3}\right)\begin{bmatrix} 3 \\ -2 \end{bmatrix} + \left(\frac{2a+3b}{3}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{Check this!}$$

11. Similar to Example 2.19, we must show $x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ can always be solved.

The augmented matrix is $\begin{bmatrix} 1 & 1 & 0 & | & a \\ 0 & 1 & 1 & | & b \\ 1 & 0 & 1 & | & c \end{bmatrix}$, and row reduction produces:

$$\begin{bmatrix} 1 & 1 & 0 & | & a \\ 0 & 1 & 1 & | & b \\ 1 & 0 & 1 & | & c \end{bmatrix} \xrightarrow{R_3 - R_1 + R_2} \begin{bmatrix} 1 & 1 & 0 & | & a \\ 0 & 1 & 1 & | & b \\ 0 & 0 & 2 & | & -a+b+c \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 1 & 0 & | & a \\ 0 & 1 & 1 & | & b \\ 0 & 0 & 1 & | & (-a+b+c)/2 \end{bmatrix}$$

$$\xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & 1 & 0 & | & a \\ 0 & 1 & 0 & | & (a+b-c)/2 \\ 0 & 0 & 1 & | & (-a+b+c)/2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & | & (a-b+c)/2 \\ 0 & 1 & 0 & | & (a+b-c)/2 \\ 0 & 0 & 1 & | & (-a+b+c)/2 \end{bmatrix}$$

We see that $x = (a-b+c)/2$, $y = (a+b-c)/2$, and $z = (-a+b+c)/2$.
So for any choice of a, b, and c we have:

$$\left(\frac{a-b+c}{2}\right)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \left(\frac{a+b-c}{2}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \left(\frac{-a+b+c}{2}\right)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{Check this!}$$

12. Similar to Example 2.19, show $x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ can always be solved.

The augmented matrix is $\begin{bmatrix} 1 & -1 & 2 & a \\ 2 & -1 & 1 & b \\ 3 & 0 & -1 & c \end{bmatrix}$, and row reduction produces:

$\begin{bmatrix} 1 & -1 & 2 & a \\ 2 & -1 & 1 & b \\ 3 & 0 & -1 & c \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 1 & -1 & 2 & a \\ 0 & 1 & -3 & -2a+b \\ 0 & 3 & -7 & -3a+c \end{bmatrix} \xrightarrow[R_3-3R_2]{R_1+R_2} \begin{bmatrix} 1 & 0 & -1 & -a+b \\ 0 & 1 & -3 & -2a+b \\ 0 & 0 & 2 & 3a-3b+c \end{bmatrix}$

$\xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & -1 & -a+b \\ 0 & 1 & -3 & -2a+b \\ 0 & 0 & 1 & (3a-3b+c)/2 \end{bmatrix} \xrightarrow[R_2+3R_3]{R_1+R_3} \begin{bmatrix} 1 & 0 & 0 & (a-b+c)/2 \\ 0 & 1 & 0 & (5a-7b+3c)/2 \\ 0 & 0 & 1 & (3a-3b+c)/2 \end{bmatrix}$

We see that $x = (a-b+c)/2$, $y = (5a-7b+3c)/2$, and $z = (3a-3b+c)/2$.
So for any choice of a, b, and c we have:

$$\left(\frac{a-b+c}{2}\right) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \left(\frac{5a-7b+3c}{2}\right) \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \left(\frac{3a-3b+c}{2}\right) \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

13. We should describe the span of the given vectors (a) geometrically and (b) algebraically.

(a) Geometrically, we can see that the set of all linear combinations of $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

is just the line through the origin with $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ as direction vector.

Why do we not have to consider $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$? Because $\begin{bmatrix} 2 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(b) Algebraically, the vector equation of this line is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} t$.

That is just another way of saying that $\begin{bmatrix} x \\ y \end{bmatrix}$ is in the span of $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Suppose we want to obtain the general equation of this line.
One method is to use the system of equations arising from the vector equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} t \Rightarrow \begin{array}{l} x = -t \\ y = 2t \end{array} \text{ So } y = 2(-x) = -2x \Rightarrow 2x + y = 0.$$

14. We should describe the span of the given vectors (a) geometrically and (b) algebraically.

  (a) Geometrically, we can see that the set of all linear combinations of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

    is just the line through the origin with $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ as direction vector.

    Why do we not have to consider $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$? Because $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

  (b) Algebraically, the vector equation of this line is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} t$.

    That is just another way of saying that $\begin{bmatrix} x \\ y \end{bmatrix}$ is in the span of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

    Suppose we want to obtain the general equation of this line.
    One method is to use the system of equations arising from the vector equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} t \Rightarrow \begin{array}{l} x = 3t \\ y = 4t \end{array} \text{ So } y = 4\left(\tfrac{1}{3}x\right) \Rightarrow 3y = 4x \Rightarrow 4x - 3y = 0.$$

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15. We should describe the span of the given vectors (a) geometrically and (b) algebraically.

  (a) Geometrically, we can see that the set of all linear combinations of $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

    is just the plane through the origin with $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ as direction vectors.

  (b) Algebraically, the vector equation of this plane is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

    That is just another way of saying that $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in the span of $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

    Suppose we want to obtain the general equation of this plane.
    One method is to use the system of equations arising from the vector equation:

$$\begin{array}{rl} s + 3t &= x \\ 2s + 2t &= y \\ -t &= z \end{array} \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & x \\ 2 & 2 & y \\ 0 & -1 & z \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & x \\ 0 & -4 & -2x + y \\ 0 & 0 & (2x - y + 4z)/4 \end{array}\right]$$

    We know this system is consistent, since $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *is in the span of* $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

    So, we *must* have $2x - y + 4z = 0$, giving us the general equation we seek.

    Note: Both $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ are orthogonal to $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$. Should they be?

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19. To show **u**, **v**, and **w** are in span(**u**, **u** + **v**, **u** + **v** + **w**), we must show that **u**, **v**, and **w** can be written as a linear combination of **u**, **u** + **v**, **u** + **v** + **w**.

  Q: Can we simply let **u** = **u**, **v** = **v**, and **w** = **w**?
  A: No. Why not? These vectors, except for **u**, are not explicitly listed in the spanning set.

    Instead, we need linear combinations of **u**, **u** + **v**, **u** + **v** + **w** that yield **u**, **v**, and **w**. So:

    u = **u**, **v** = −**u** + (**u** + **v**), and **w** = −(**u** + **v**) + (**u** + **v** + **w**).

  Note: We have now shown that we can use **u**, **v**, and **w**. How?

(optional)

21. When proving something for n, first let $n = 1$ or 2 to look for the underlying pattern.

Assume that there are only two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ and two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
We are told $\mathbf{w}$ is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. So: $\mathbf{w} = w_1\mathbf{u}_1 + w_2\mathbf{u}_2$.
We are also told that both $\mathbf{u}_1$ and $\mathbf{u}_2$ are linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.
So, we have both: $\mathbf{u}_1 = v_{11}\mathbf{v}_1 + v_{12}\mathbf{v}_2$ and $\mathbf{u}_2 = v_{21}\mathbf{v}_1 + v_{22}\mathbf{v}_2$.
We need to show these assumptions imply $\mathbf{w}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. How?
Let $\mathbf{u}_1 = u_{1_1}\mathbf{v}_1 + u_{1_2}\mathbf{v}_2$ and $\mathbf{u}_2 = u_{2_1}\mathbf{v}_1 + u_{2_2}\mathbf{v}_2$ in $\mathbf{w} = w_1\mathbf{u}_1 + w_2\mathbf{u}_2$.
This substitution yields: $\mathbf{w} = w_1(u_{1_1}\mathbf{v}_1 + u_{1_2}\mathbf{v}_2) + w_2(u_{2_1}\mathbf{v}_1 + u_{2_2}\mathbf{v}_2)$.
It is now obvious that $\mathbf{w}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Why?
Observe that this reasoning holds for any n and proceed to the proof.

(a) Let $\mathbf{w} = w_1\mathbf{u}_1 + w_1\mathbf{u}_2 + \cdots + w_1\mathbf{u}_k$,
and assume that each $\mathbf{u}_i$ is a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$.
Then each $\mathbf{u}_i = u_{i_1}\mathbf{v}_1 + u_{i_2}\mathbf{v}_2 + \cdots + u_{i_m}\mathbf{v}_m$, and

$$\begin{aligned}
\mathbf{w} &= w_1\mathbf{u}_1 + w_2\mathbf{u}_2 + \cdots + w_k\mathbf{u}_k \\
&= w_1(u_{1_1}\mathbf{v}_1 + u_{1_2}\mathbf{v}_2 + \cdots + u_{1_m}\mathbf{v}_m) + w_2(u_{2_1}\mathbf{v}_1 + u_{2_2}\mathbf{v}_2 + \cdots + u_{2_m}\mathbf{v}_m) + \cdots \\
&\qquad \cdots + w_k(u_{k_1}\mathbf{v}_1 + u_{k_2}\mathbf{v}_2 + \cdots + u_{k_m}\mathbf{v}_m) \\
&= (w_1u_{1_1} + w_2u_{2_1} + \cdots + w_ku_{k_1})\mathbf{v}_1 + (w_1u_{1_2} + w_2u_{2_2} + \cdots + w_ku_{k_2})\mathbf{v}_2 + \cdots \\
&\qquad \cdots + (w_1u_{1_m} + w_2u_{2_m} + \cdots + w_ku_{k_m})\mathbf{v}_m \\
&= w_1'\mathbf{v}_1 + w_2'\mathbf{v}_2 + \cdots + w_m'\mathbf{v}_m.
\end{aligned}$$

So, any vector $\mathbf{w} \in \text{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k)$ is also in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$,
and $\text{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k) \subseteq \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$

(b) Suppose that in addition to (a), each $\mathbf{v}_j$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$.
Let $\mathbf{w}$ be an arbitrary vector in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$.
Then $\mathbf{w} = w_1'\mathbf{v}_1 + w_2'\mathbf{v}_2 + \cdots + w_m'\mathbf{v}_m$, but each $\mathbf{v}_j = v_{j_1}\mathbf{u}_1 + v_{j_2}\mathbf{u}_2 + \cdots + v_{j_k}\mathbf{u}_k$, so

$$\begin{aligned}
\mathbf{w} &= w_1'(v_{1_1}\mathbf{u}_1 + v_{1_2}\mathbf{u}_2 + \cdots + v_{1_k}\mathbf{u}_k) + w_2'(v_{2_1}\mathbf{u}_1 + v_{2_2}\mathbf{u}_2 + \cdots + v_{2_k}\mathbf{u}_k) + \cdots \\
&\qquad \cdots + w_m'(v_{m_1}\mathbf{u}_1 + v_{m_2}\mathbf{u}_2 + \cdots + v_{m_k}\mathbf{u}_k) \\
&= (w_1'v_{1_1} + w_2'v_{2_1} + \cdots + w_m'v_{m_1})\mathbf{u}_1 + (w_1'v_{1_2} + w_2'v_{2_2} + \cdots + w_m'v_{m_2})\mathbf{u}_2 + \cdots \\
&\qquad \cdots + (w_1'v_{1_k} + w_2'v_{2_k} + \cdots + w_m'v_{m_k})\mathbf{u}_k \\
&= w_1\mathbf{u}_1 + w_1\mathbf{u}_2 + \cdots + w_k\mathbf{u}_k.
\end{aligned}$$

So, any vector $\mathbf{w} \in \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ is also in $\text{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k)$,
and $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m) \subseteq \text{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k)$.
But we already had $\text{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k) \subseteq \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$,
so $\text{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k) = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$.

(c) Need only show $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are linear combinations of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

That's obvious since $\mathbf{e}_1 = \mathbf{v}_1$, $\mathbf{e}_2 = \mathbf{v}_2 - \mathbf{v}_1$, and $\mathbf{e}_3 = \mathbf{v}_3 - \mathbf{v}_2$. Why is that enough?
Because then we have $\mathbb{R}^3 = \text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

22. The vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}$ are linearly independent.

This can be determined by inspection because they are not scalar multiples of each other.

23. Since there is no obvious dependence relation here, we follow Example 2.23.

Find scalars c_1, c_2, and c_3 such that: $c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Form the linear system, its associated augmented matrix, and row reduce to solve:

$$\begin{matrix} c_1 + c_2 + c_3 = 0 \\ c_1 + 2c_2 - c_3 = 0 \\ c_1 + 3c_2 + 2c_3 = 0 \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since $c_1 = c_2 = c_3 = 0$ is the unique solution, the vectors are linearly independent.

24. Since there is no obvious dependence relation here, we follow Example 2.23.

Find scalars c_1, c_2, and c_3 such that: $c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Form the linear system, its associated augmented matrix, and row reduce to solve:

$$\begin{matrix} 2c_1 + 3c_2 + c_3 = 0 \\ 2c_1 + c_2 - 5c_3 = 0 \\ c_1 + 2c_2 + 2c_3 = 0 \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 2 & 1 & -5 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since $c_1 = 4c_3$ and $c_2 = -3c_3$ is a solution, the vectors are linearly dependent.

One dependence relationship is: $4 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Are there others?

43. We apply *(optional)* the definition of linear independence and Examples 2.23 and 2.25 to prove our claims.

(a) We will show that $\mathbf{u} + \mathbf{v}$, $\mathbf{v} + \mathbf{w}$, and $\mathbf{u} + \mathbf{w}$ are linearly independent.
Given $c_1(\mathbf{u} + \mathbf{v}) + c_2(\mathbf{v} + \mathbf{w}) + c_3(\mathbf{u} + \mathbf{w}) = 0$, we will show $c_1 = c_2 = c_3 = 0$.
Multiplying and gathering like terms yields: $(c_1 + c_3)\mathbf{u} + (c_1 + c_2)\mathbf{v} + (c_2 + c_3)\mathbf{w} = 0$.
Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, $c_1 + c_3 = c_1 + c_2 = c_2 + c_3 = 0$.
We create the matrix of coefficients A and row reduce to determine its rank:

$$\begin{matrix} c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \\ c_2 + c_3 = 0 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since $\text{rank}(A) = 3$ the only solution is the trivial one, so $c_1 = c_2 = c_3 = 0$.

(b) We will show that $\mathbf{u} - \mathbf{v}$, $\mathbf{v} - \mathbf{w}$, and $\mathbf{u} - \mathbf{w}$ are linearly dependent.
Given $c_1(\mathbf{u} - \mathbf{v}) + c_2(\mathbf{v} - \mathbf{w}) + c_3(\mathbf{u} - \mathbf{w}) = 0$, we will show $c_1 = c_2 = -c_3$.
Multiplying and gathering like terms yields: $(c_1 + c_3)\mathbf{u} + (-c_1 + c_2)\mathbf{v} + (-c_2 - c_3)\mathbf{w} = 0$.
Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, $c_1 + c_3 = -c_1 + c_2 = -c_2 - c_3 = 0$.
We form the augmented matrix and row reduce to solve:

$$\begin{matrix} c_1 + c_3 = 0 \\ -c_1 + c_2 = 0 \\ -c_2 - c_3 = 0 \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This clearly has the solution $c_1 = c_2 = -c_3$ as we were to show.

25. The vectors $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ are linearly dependent.

This can be determined by inspection because $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$.

30. The vectors $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ are linearly independent.

This can be determined by inspection. How?

To create a 0 in the first component, the coefficient of $\mathbf{v}_4$ must be 0. Why?

Given that, a 0 in the second component forces the coefficient of $\mathbf{v}_3$ to be 0.

Given those two facts, a 0 in the third component forces the coefficient of $\mathbf{v}_2$ to be 0.

And finally, given all that, a 0 in the fourth component forces the coefficient of $\mathbf{v}_1$ to be 0.

To verify this argument, we follow Example 2.23.

Find scalars c_1, c_2, c_3, and c_4 such that: $c_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Form the linear system, its associated augmented matrix, and row reduce to solve:

$$\begin{array}{rrrrl} & & & c_4 & = 0 \\ & & 2c_3 & + \ c_4 & = 0 \\ & 3c_2 & + \ 2c_3 & + \ c_4 & = 0 \\ 4c_1 & + \ 3c_2 & + \ 2c_3 & + \ c_4 & = 0 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Since $c_1 = c_2 = c_3 = c_4 = 0$ is the unique solution, the vectors are linearly independent.

31. The vectors $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix}$ are linearly dependent.

This can be determined by inspection because $\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$.

34. Exercises 32 through 41 provide a check on our solutions to Exercises 22 through 31. How? In these exercises the directions tell us to follow Example 2.25 and apply Theorem 2.7:

We construct a matrix with these vectors as its rows and proceed to reduce it to echelon form. Each time a row changes, we denote the new row by adding a prime symbol:

$$A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & -5 & 2 \end{bmatrix} \begin{smallmatrix} R_1'=R_3 \\ R_2'=R_1 \\ R_3'=R_2 \end{smallmatrix} \begin{bmatrix} 1 & -5 & 2 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{smallmatrix} R_2''=R_2'-2R_1' \\ R_3''=R_3'-3R_1' \end{smallmatrix} \begin{bmatrix} 1 & -5 & 2 \\ 0 & 12 & -3 \\ 0 & 16 & -4 \end{bmatrix} \begin{smallmatrix} R_3'''=R_3''-\frac{4}{3}R_2'' \end{smallmatrix} \begin{bmatrix} 1 & -5 & 2 \\ 0 & 12 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

We can stop. Why? We have created a zero row. What does that tell us?
Since the rank of a matrix is the number of nonzero rows in its row echelon form, $\text{rank}(A) = 2$. What do we conclude? We conclude $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent. How?

Theorem 2.7 states that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if $\text{rank}(A) < m$. So, since $\text{rank}(A) = 2 < 3$, Theorem 2.7 implies $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent.

Furthermore, from the row reduction above, we see that: $\mathbf{0} = R_3''' = R_3'' - \frac{4}{3}R_2''$.

Multiplying both sides by 3 implies: $\mathbf{0} = 3R_3'' - 4R_2''$. Substituting, we have:

$\mathbf{0} = 3(R_3' - 3R_1') - 4(R_2' - 2R_1') = 3(R_2 - 3R_3) - 4(R_1 - 2R_3) = -4R_1 + 3R_2 - R_3$.

Multiplying both sides by -1 yields this dependence relation among the original vectors:

$$4\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ Compare this result to Exercise 24. Does it agree?

35. Exercises 32 through 41 provide a check on our solutions to Exercises 22 through 31. How? In these exercises the directions tell us to follow Example 2.25 and apply Theorem 2.7:

We construct a matrix with these vectors as its rows and proceed to reduce it to echelon form. Each time a row changes, we denote the new row by adding a prime symbol:

$$A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 0 & 1 \end{bmatrix} \begin{smallmatrix} R_1'=R_2 \\ R_2'=R_1 \end{smallmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{smallmatrix} R_3'=R_3-R_1' \end{smallmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \begin{smallmatrix} R_3''=R_3'+R_2' \end{smallmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

We can stop. Why? We have created a zero row. What does that tell us?
Since the rank of a matrix is the number of nonzero rows in its row echelon form, $\text{rank}(A) = 2$. What do we conclude? We conclude $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent. How?

Theorem 2.7 states that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if $\text{rank}(A) < m$. So, since $\text{rank}(A) = 2 < 3$, Theorem 2.7 implies $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent.

From the row reduction, we see: $\mathbf{0} = R_3'' = R_3' + R_2' = (R_3' - R_1') + R_2' = R_3 - R_2 + R_1$.

This equation yields a dependence relation among the original vectors:

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ Compare this result to Exercise 25. Does it agree?

(optional)

38. Exercises 32 through 41 provide a check on our solutions to Exercises 22 through 31. How?
In these exercises the directions tell us to follow Example 2.25 and apply Theorem 2.7:

We construct a matrix with these vectors as its rows and proceed to reduce it to echelon form.
Each time a row changes, we denote the new row by adding a prime symbol:

$$A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 & 1 \\ 3 & 2 & 2 & 4 \\ 2 & 3 & 1 & -1 \end{bmatrix} \xrightarrow[R_3'=R_3+2R_1]{R_2'=R_2+3R_1} \begin{bmatrix} -1 & 1 & 2 & 1 \\ 0 & 5 & 8 & 7 \\ 0 & 5 & 5 & 1 \end{bmatrix} \xrightarrow{R_3''=R_3'-R_2'} \begin{bmatrix} -1 & 1 & 2 & 1 \\ 0 & 5 & 8 & 7 \\ 0 & 0 & -3 & -6 \end{bmatrix}$$

We can stop. Why? We have put A into row echelon form. How can we tell?
Since the rank of a matrix is the number of nonzero rows in its row echelon form, $\text{rank}(A) = 3$.
What do we conclude? We conclude $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent. How?

Theorem 2.7 states that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if $\text{rank}(A) < m$.
But that implies the following: If $\text{rank}(A) \geq m$, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent.

In this case, therefore, we argue as follows:
Since $\text{rank}(A) \geq 3$, Theorem 2.7 implies $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

Does the agree with the solution we found in Exercise 28? It should.
Which method was easier for this Exercise? Why?

① $z_1 + z_2 = 3 + 1.7i + (-4 + 2i)$

$= -1 + 3.7i$

② $z_1 z_2 = (-2 + 5i)(11 - 3i)$

$= -22 + 15 + i(5 \cdot 11 + 6)$

$= -7 + 61i$

③ $e^{5+7i} = e^5(\cos 7 + i \sin 7)$

$= e^5 \cos 7 + i e^5 \sin 7$

④ (a) $\dfrac{1}{1+i} = \dfrac{1-i}{(1-i)(1+i)} = \dfrac{1}{2} - \dfrac{i}{2}$

(b) $\dfrac{2+i}{3+2i} = \dfrac{(2+i)(3-2i)}{(3+2i)(3-2i)} = \dfrac{6 + 2 + i(3-4)}{13}$

$= \dfrac{8}{13} - \dfrac{i}{13}$

(c) $e^{1+\pi i} = e^1(\cos \pi + i\sin \pi)$

$$= -e^1$$

d) $e^{-\pi i} = \cos(-\pi) + i\sin(-\pi) = -1$

Ⓢ $z = 7 + 10i$

$$r = \sqrt{7^2 + 10^2} = \sqrt{149}$$

$\varphi = \arg z = \arctan \frac{10}{7}$

$$z = \sqrt{149}\, e^{i \arctan \frac{10}{7}}$$

1. Following Examples 3.1 through 3.5, we have:

$$A + 2D = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} + 2\begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3+2(0) & 0+2(-3) \\ -1+2(-2) & 5+2(1) \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ -5 & 7 \end{bmatrix}.$$

2. Following Examples 3.1 through 3.5, we have:

$$3D - 2A = 3\begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} - 2\begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3(0)-2(3) & 3(-3)-2(0) \\ 3(-2)-2(-1) & 3(1)-2(5) \end{bmatrix} = \begin{bmatrix} -6 & -9 \\ -4 & -7 \end{bmatrix}.$$

3. $B - C$ is not possible. Why not? B is a 2×3 matrix and C is a 3×2 matrix. We can only add and subtract matrices of the same size.

4. Applying the definition the transpose to C^T, the transpose of matrix C, we have:

Since $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $C^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, $B - C^T = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -5 & -4 \\ -2 & -2 & -3 \end{bmatrix}.$

5. By the definition of the matrix product, $C = AB$, and Example 3.6, we have: $AB =$

$$\begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3(4)+0(0) & 3(-2)+0(2) & 3(1)+0(3) \\ (-1)(4)+5(0) & 3(-2)+(-1)(2) & 3(1)+(-1)(3) \end{bmatrix} = \begin{bmatrix} 12 & -6 & 3 \\ -4 & 12 & 14 \end{bmatrix}.$$

6. From the remarks following the definition of matrix multiplication, we see BD is not possible. Why not? B is a 2×3 matrix and D is a 2×2 matrix. What does that tell us? The number of columns in $B = 3 \neq 2 =$ the number of rows in D. For matrix multiplication, the number of columns in B has to equal the number of rows in D.

In the future, when checking whether or not matrix multiplication is possible, we will write: B is $[2 \times 3]$ and D is $[2 \times 2]$, so BD is $[2 \times 3][2 \times 2]$ which is not possible because the *inner* numbers do not match.

Q: Do they outer numbers have to match? If not, what do they tell us?
A: No, they do not have to match. The *outer* numbers tell us the size, $[r \times c]$, of the result.

7. We begin by applying the definition of matrix multiplication to see if BC is possible:
Since B is $[2 \times 3]$ and C is $[3 \times 2]$, BC, $[2 \times 3][3 \times 2]$, is possible. Why?
Because the *inner* numbers match. What does that tell us?
The number of columns in $B = 3 =$ the number of rows in C.

Furthermore, since BC is $[2 \times 3][3 \times 2]$, BC will be a 2×2 matrix.
Since D is also a 2×2 matrix, we can add them together. That is, $D + BC$ is possible.

$$\text{First, } BC = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 4(1)-2(3)+1(5) & 4(2)-2(4)+1(6) \\ 0(1)+2(3)+3(5) & 0(2)+2(4)+3(6) \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 21 & 26 \end{bmatrix}.$$

$$\text{So, } D + BC = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 21 & 26 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 19 & 27 \end{bmatrix}.$$

8. We should note that $B^T B$ is *always* possible. Why?
the number of columns of B^T = the number of rows of B (by the definition of B^T).

Since $B = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}$ and $B^T = \begin{bmatrix} 4 & 0 \\ -2 & 2 \\ 1 & 3 \end{bmatrix}$,

$$B^T B = \begin{bmatrix} 4 & 0 \\ -2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4(4)+0(0) & 4(-2)+0(2) & 4(1)+0(3) \\ (-2)(4)+2(0) & (-2)(-2)+2(2) & (-2)(1)+2(3) \\ 1(4)+3(0) & 1(-2)+3(2) & 1(1)+3(3) \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -8 & 4 \\ -8 & 8 & 4 \\ 4 & 4 & 10 \end{bmatrix}.$$ Is BB^T always possible as well? Why or why not?

9. Before we begin, we should determine if AF and $E(AF)$ are possible.
Since A is $[2 \times 2]$ and F is $[2 \times 1]$, AF, $[2 \times 2][2 \times 1]$, is possible. Why?
Because the *inner* numbers match. What does that tell us?
The number of columns in $A = 2 =$ the number of rows in F.
Furthermore, since AF is $[2 \times 2][2 \times 1]$, AF will be a 2×1 matrix.

Since E is $[1 \times 2]$ and AF is $[2 \times 1]$, $E(AF)$, $[1 \times 2][2 \times 1]$, is possible. Why?
Because the *inner* numbers match. What does that tell us?
The number of columns in $E = 1 =$ the number of rows in AF.
Furthermore, since $E(AF)$ is $[1 \times 2][2 \times 1]$, $E(AF)$ will be a 1×1 matrix.

First, $AF = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3(-1)+0(2) \\ (-1)(-1)+5(2) \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \end{bmatrix}.$

So $E(AF) = [4\ 2] \begin{bmatrix} -3 \\ 11 \end{bmatrix} = [4(-3)+2(11)] = [10].$

13. $B^T C^T - (CB)^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

14. $DA - AD = \begin{bmatrix} 3 & -6 \\ 3 & -3 \end{bmatrix}.$

19. The number of units of each product shipped to each warehouse is given by $A = \begin{bmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{bmatrix}.$

The cost of shipping one unit of each product is given by $B = \begin{bmatrix} 1.50 & 1.00 & 2.00 \\ 1.75 & 1.50 & 1.00 \end{bmatrix}$

(where b_{ij} is the cost of shipping a unit of product j by $i = 1$ truck, $i = 2$ train).
Compare the cost of shipping the products to each of the warehouses:

$$BA = \begin{bmatrix} 1.50 & 1.00 & 2.00 \\ 1.75 & 1.50 & 1.00 \end{bmatrix} \begin{bmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{bmatrix} = \begin{bmatrix} 650.00 & 462.50 \\ 675.00 & 406.25 \end{bmatrix} \Rightarrow$$

It is cheaper to ship the products to warehouse 1 by truck, but to warehouse 2 by train.

21. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$

23. $A\mathbf{b}_1 = 2\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 5 \end{bmatrix}$, $A\mathbf{b}_2 = 3\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -9 \\ -4 \\ 0 \end{bmatrix}$,

and $A\mathbf{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -8 \\ 5 \\ -4 \end{bmatrix}$. Therefore, $AB = \begin{bmatrix} 4 & -9 & -8 \\ -6 & -4 & 5 \\ 5 & 0 & -4 \end{bmatrix}$.

24. $\mathbf{A}_1 B = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix} - 2\begin{bmatrix} -1 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -9 & -8 \end{bmatrix}$,

$\mathbf{A}_2 B = -3\begin{bmatrix} 2 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 6 & 4 \end{bmatrix} = \begin{bmatrix} -6 & -4 & 5 \end{bmatrix}$, and

$\mathbf{A}_3 B = 2\begin{bmatrix} 2 & 3 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & -4 \end{bmatrix}$. Thus, $AB = \begin{bmatrix} 4 & -9 & -8 \\ -6 & -4 & 5 \\ 5 & 0 & -4 \end{bmatrix}$.