HW 5 Solutions

Math 328

3.1

10. Before we begin, we should determine if DF and $F(DF)$ are possible.

Since D is $[2 \times 2]$ and F is $[2 \times 1]$, DF, $[2 \times 2][2 \times 1]$, is possible. Why?
Because the *inner* numbers match. What does that tell us?
The number of columns in $D = 2 =$ the number of rows in F.
Furthermore, since DF is $[2 \times 2][2 \times 1]$, DF will be a 2×1 matrix.

Since F is $[2 \times 1]$ and DF is $[2 \times 1]$, $F(DF)$, $[2 \times 1][2 \times 1]$, is not possible. Why?
Because the *inner* numbers do not match. What does that tell us?
The number of columns in $F = 1 \neq 2 =$ the number of rows in DF.

So, $F(DF)$ is not possible.

11. Since FE, $[2 \times 1][1 \times 2]$, is possible and yields a 2×2 matrix, we have:

$$FE = \begin{bmatrix} -1 \\ 2 \end{bmatrix} [4 \;\; 2] = \begin{bmatrix} -1(4) & (-1)(2) \\ 2(4) & 2(2) \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 8 & 4 \end{bmatrix}.$$

Q: Does FE equal EF?
A: No. In fact, note that EF, $[1 \times 2][2 \times 1]$, is possible and yields a 1×1 matrix.
This is a good example of the general fact that matrix multiplication does not commut

12. Since EF, $[1 \times 2][2 \times 1]$, is possible and yields a 1×1 matrix, we have:

$$EF = [4 \;\; 2] \begin{bmatrix} -1 \\ 2 \end{bmatrix} = [4(-1) + 2(2)] = [0].$$

Since Exercises 13 through 16 use the skills above, we simply present the answers below.

15. $A^3 = \begin{bmatrix} 27 & 0 \\ -49 & 125 \end{bmatrix}.$

25. The outer product expansion of AB is

$$\mathbf{a}_1\mathbf{B}_1 + \mathbf{a}_2\mathbf{B}_2 + \mathbf{a}_3\mathbf{B}_3 = \begin{bmatrix} 2 & 3 & 0 \\ -6 & -9 & 0 \\ 4 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -12 & -8 \\ -1 & 6 & 4 \\ 1 & -6 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -9 & -8 \\ -6 & -4 & 5 \\ 5 & 0 & -4 \end{bmatrix}.$$

30. Let $A = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix}$ and $AB = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_n B \end{bmatrix}$. Then assume there exist x_i not all zero such that:

$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$, that is, the rows of A are linearly dependent.

So, we have: $(x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n)\, B = x_1(\mathbf{a}_1 B) + x_2(\mathbf{a}_2 B) + \cdots + x_n(\mathbf{a}_n B) = 0 \Rightarrow$
The rows of AB are linearly dependent.

31. For matrices A, B we have the block structure $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \Rightarrow$

$$\begin{aligned}
AB &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 3 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.
\end{aligned}$$

34. $AB = \begin{bmatrix} 2 & 3 & 4 & 0 \\ 2 & 3 & 6 & -1 \\ 3 & 3 & 4 & -2 \\ 4 & 4 & 4 & -4 \end{bmatrix}.$

35. (a) Computing the powers of matrix A as required, we have:

$$A^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix},\ A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ A^4 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix},$$

$$A^5 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix},\ A^6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A^7 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = A!$$

(b) From our work in (a), we see that $A^1 = A^7 = A^{1 \cdot 6 + 1}$.

So the powers of A that actually create *distinct* matrices act like $\mathbb{Z}^6$.

See Section 1.4, Examples 1.32 through 1.35.

So, to determine A^{2001}, we should first determine the value of 2001 in $\mathbb{Z}^6$.

How? Divide 2001 by 6 and look at the remainder: $2001 = 333 \cdot 6 + 3 = 3$ in $\mathbb{Z}^6$.

Therefore $A^{2001} = A^{333 \cdot 6 + 3} = A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$

(optional)

3

38. We will prove (b) using *induction*. See Appendix B for discussion and examples. We make use of the following trigonometric identities in our proof below: $\cos\theta\cos n\theta - \sin\theta\sin n\theta = \cos(n+1)\theta$ and $\sin\theta\cos n\theta + \cos\theta\sin n\theta = \sin(n+1)\theta$.

(a) $A^2 = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}\cos^2\theta-\sin^2\theta & -2\cos\theta\sin\theta\\ 2\cos\theta\sin\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}$.

But $\cos^2\theta - \sin^2\theta = \cos 2\theta$, and $2\cos\theta\sin\theta = \sin 2\theta$, so $A^2 = \begin{bmatrix}\cos 2\theta & -\sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{bmatrix}$.

(b) We will show that $A^n = \begin{bmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{bmatrix}$ for $n \geq 1$ by induction.

1: $A^1 = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$. This is obvious, so there is nothing to show.

n: $A^n = \begin{bmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{bmatrix}$. This is the induction hypothesis.

$n+1$: $A^{n+1} = \begin{bmatrix}\cos(n+1)\theta & -\sin(n+1)\theta\\ \sin(n+1)\theta & \cos(n+1)\theta\end{bmatrix}$.

This is the statement we must prove using the induction hypothesis.

$$\begin{aligned} A^{n+1} = A^1A^n &\overset{\substack{\text{by}\\\text{induction}}}{=} \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{bmatrix}\\ &\overset{\substack{\text{by matrix}\\\text{multiplication}}}{=} \begin{bmatrix}\cos\theta\cos n\theta - \sin\theta\sin n\theta & -(\cos\theta\sin n\theta + \sin\theta\cos n\theta)\\ \sin\theta\cos n\theta + \cos\theta\sin n\theta & \cos\theta\cos n\theta - \sin\theta\sin n\theta\end{bmatrix}\\ &\overset{\substack{\text{by trigonometric}\\\text{identities}}}{=} \begin{bmatrix}\cos(n+1)\theta & -\sin(n+1)\theta\\ \sin(n+1)\theta & \cos(n+1)\theta\end{bmatrix}\end{aligned}$$

We have shown (by induction) that $A^n = \begin{bmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{bmatrix}$ for $n \geq 1$.

3.2

2. Following remarks prior to Example 3.16, the key assumption is matrices are the same size. Then add, subtract, and multiply (by scalars only) as in *normal* algebra.

$$2X = A - B \Rightarrow X = \tfrac{1}{2}(A-B) = \begin{bmatrix}1 & 1\\ 1 & \frac{3}{2}\end{bmatrix}.$$

6. As in Example 3.16, we form the augmented matrix and row reduce to solve. As in Exercise 5, the first column is the entries of A_1, the second column is the entries of A_2, the third column is the entries of A_3, and the augmented column is the entries of B.

$$\left[\begin{array}{ccc|c}1 & 0 & 1 & 2\\ 0 & -1 & 1 & 3\\ 0 & 1 & 0 & -4\\ 1 & 0 & 1 & 2\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1 & 0 & 0 & 3\\ 0 & 1 & 0 & -4\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{array}\right] \Rightarrow \quad B = 3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - 4\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} - \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}.$$

14. Following Example 3.18, we create an augmented matrix and row reduce to solve.
As in Exercise 8, the first column is the entries of A_1, the second column is the entries of A_2, the third column is the entries of A_3, but now the augmented column is all zeroes.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, $c_1 = -3c_3$, $c_2 = c_3$ is a solution with at least one $c_i \neq 0$. What does that tell us?
That tells us that A_1, A_2, and A_3 are linearly dependent.
In particular, if we let $c_3 = -1$, we have the following dependence relation:

$$3\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

18. (e)
$$
\begin{aligned}
c(A+B) &= c\left(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}\right) \\
&= c\left(\begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix}\right) \\
&= \begin{bmatrix} c(a_{11}+b_{11}) & \cdots & c(a_{1n}+b_{1n}) \\ \vdots & \ddots & \vdots \\ c(a_{m1}+b_{m1}) & \cdots & c(a_{mn}+b_{mn}) \end{bmatrix} \\
&= \begin{bmatrix} ca_{11}+cb_{11} & \cdots & ca_{1n}+cb_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1}+cb_{m1} & \cdots & ca_{mn}+cb_{mn} \end{bmatrix} \\
&= \begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{bmatrix} + \begin{bmatrix} cb_{11} & \cdots & cb_{1n} \\ \vdots & \ddots & \vdots \\ cb_{m1} & \cdots & cb_{mn} \end{bmatrix} = cA + cB.
\end{aligned}
$$

(f)
$$
\begin{aligned}
(c+d)A &= (c+d)\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} (c+d)a_{11} & \cdots & (c+d)a_{1n} \\ \vdots & \ddots & \vdots \\ (c+d)a_{m1} & \cdots & (c+d)a_{mn} \end{bmatrix} \\
&= \begin{bmatrix} ca_{11}+da_{11} & \cdots & ca_{1n}+da_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1}+da_{m1} & \cdots & ca_{mn}+da_{mn} \end{bmatrix} \\
&= \begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{bmatrix} + \begin{bmatrix} da_{11} & \cdots & da_{1n} \\ \vdots & \ddots & \vdots \\ da_{m1} & \cdots & da_{mn} \end{bmatrix} \\
&= cA + dA.
\end{aligned}
$$

(g)
$$
c(dA) = c\begin{bmatrix} da_{11} & \cdots & da_{1n} \\ \vdots & \ddots & \vdots \\ da_{m1} & \cdots & da_{mn} \end{bmatrix} = \begin{bmatrix} cda_{11} & \cdots & cda_{1n} \\ \vdots & \ddots & \vdots \\ cda_{m1} & \cdots & cda_{mn} \end{bmatrix} = (cd)\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = (cd)A.
$$

(h)
$$
1A = 1\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} 1a_{11} & \cdots & 1a_{1n} \\ \vdots & \ddots & \vdots \\ 1a_{m1} & \cdots & 1a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = A.
$$

19. Let A, B, and C be matrices of appropriate dimensions. Then

$$(A+B)C = \left(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} \right) \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}+b_{11})\,c_{11} + \cdots + (a_{m1}+b_{m1})\,c_{1n} & \cdots & (a_{1n}+b_{1n})\,c_{11} + \cdots + (a_{mn}+b_{mn})\,c_{1n} \\ \vdots & \ddots & \vdots \\ (a_{11}+b_{11})\,c_{m1} + \cdots + (a_{m1}+b_{m1})\,c_{mn} & \cdots & (a_{1n}+b_{1n})\,c_{m1} + \cdots + (a_{mn}+b_{mn})\,c_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}c_{11} + b_{11}c_{11} + \cdots + a_{m1}c_{1n} + b_{m1}c_{1n} & \cdots & a_{1n}c_{11} + b_{1n}c_{11} + \cdots + a_{mn}c_{1n} + b_{mn}c_{1n} \\ \vdots & \ddots & \vdots \\ a_{11}c_{m1} + b_{11}c_{m1} + \cdots + a_{m1}c_{mn} + b_{m1}c_{mn} & \cdots & a_{1n}c_{m1} + b_{1n}c_{m1} + \cdots + a_{mn}c_{mn} + b_{mn}c_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{array}{l} (a_{11}c_{11} + \cdots + a_{m1}c_{1n}) + \\ \quad + (b_{11}c_{11} + \cdots + b_{m1}c_{1n}) \end{array} & \cdots & \begin{array}{l} (a_{1n}c_{11} + \cdots + a_{mn}c_{1n}) + \\ \quad + (b_{1n}c_{11} + \cdots + b_{mn}c_{1n}) \end{array} \\ \vdots & \ddots & \vdots \\ \begin{array}{l} (a_{11}c_{m1} + \cdots + a_{m1}c_{mn}) + \\ \quad + (b_{11}c_{m1} + \cdots + b_{m1}c_{mn}) \end{array} & \cdots & \\ \begin{array}{l} (a_{1n}c_{m1} + \ldots + +a_{mn}c_{mn}) + \\ \quad + (b_{1n}c_{m1} + \cdots + b_{mn}c_{mn}) \end{array} & & \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}c_{11} + \cdots + a_{m1}c_{1n}) & \cdots & (a_{1n}c_{11} + \cdots + a_{mn}c_{1n}) \\ \vdots & \ddots & \vdots \\ (a_{11}c_{m1} + \cdots + a_{m1}c_{mn}) & \cdots & (a_{1n}c_{m1} + \cdots + +a_{mn}c_{mn}) \end{bmatrix}$$

$$+ \begin{bmatrix} (b_{11}c_{11} + \cdots + b_{m1}c_{1n}) & \cdots & (b_{1n}c_{11} + \cdots b_{mn}c_{1n}) \\ \vdots & \ddots & \vdots \\ (b_{11}c_{m1} + \cdots + b_{m1}c_{mn}) & \cdots & (b_{1n}c_{m1} + \cdots b_{mn}c_{mn}) \end{bmatrix}$$

$$= AC + BC.$$

20. Let A, and B be matrices of appropriate dimensions, and let k be a scalar. Then

$$
\begin{aligned}
k\,(AB) &= k\left(\begin{bmatrix} a_{11}b_{11}+\cdots+a_{1n}b_{m1} & \cdots & a_{11}b_{1n}+\cdots+a_{1n}b_{mn} \\ \vdots & \ddots & \vdots \\ a_{m1}b_{11}+\cdots+a_{mn}b_{m1} & \cdots & a_{m1}b_{1n}+\cdots+a_{mn}b_{mn} \end{bmatrix}\right) \\
&= \begin{bmatrix} k\,(a_{11}b_{11}+\cdots+a_{1n}b_{m1}) & \cdots & k\,(a_{11}b_{1n}+\cdots+a_{1n}b_{mn}) \\ \vdots & \ddots & \vdots \\ k\,(a_{m1}b_{11}+\cdots+a_{mn}b_{m1}) & \cdots & k\,(a_{m1}b_{1n}+\cdots+a_{mn}b_{mn}) \end{bmatrix} \\
&= \begin{bmatrix} (ka_{11})\,b_{11}+\cdots+(ka_{1n})\,b_{m1} & \cdots & (ka_{11})\,b_{1n}+\cdots+(ka_{1n})\,b_{mn} \\ \vdots & \ddots & \vdots \\ (ka_{m1})\,b_{11}+\cdots+(ka_{mn})\,b_{m1} & \cdots & (ka_{m1})\,b_{1n}+\cdots+(ka_{mn})\,b_{mn} \end{bmatrix} \\
&= \begin{bmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{m1} & \cdots & ka_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} \\
&= (kA)\,B.
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{bmatrix} a_{11}\,(kb_{11})+\cdots+a_{1n}\,(kb_{m1}) & \cdots & a_{11}\,(kb_{1n})+\cdots+a_{1n}\,(kb_{mn}) \\ \vdots & \ddots & \vdots \\ a_{m1}\,(kb_{11})+\cdots+a_{mn}\,(kb_{m1}) & \cdots & a_{m1}\,(kb_{1n})+\cdots+a_{mn}\,(kb_{mn}) \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} kb_{11} & \cdots & kb_{1n} \\ \vdots & \ddots & \vdots \\ kb_{m1} & \cdots & kb_{mn} \end{bmatrix} \\
&= A\,(kB)\,.
\end{aligned}
$$

22. We need to show $AB = BA$ if and only if $(A-B)(A+B) = A^2 + B^2$.

Note: $(A-B)(A+B) \overset{\substack{\text{left}\\ \text{distributivity}}}{=} (A-B)A + (A-B)B \overset{\substack{\text{right}\\ \text{distributivity}}}{=} A^2 - BA + AB - B^2.$

If $AB = BA$, then $-BA + AB = O$,

so $(A-B)(A+B) = A^2 - BA + AB - B^2 \overset{\substack{\text{because}\\ -BA+AB=O}}{=} A^2 - B^2.$

If $(A-B)(A+B) = A^2 - BA + AB - B^2 = A^2 - B^2$,
then $-BA + AB = O$ so $AB = BA$.

30. Let A, B be matrices whose sizes permit the indicated operations and let k be a scalar. Denote the ith row of a matrix X by $\text{row}_i(X)$ and its jth column by $\text{col}_j(X)$.

Theorem 3.4(a): $\left[\left(A^T\right)^T\right]_{ij} = \left[A^T\right]_{ji} = [A]_{ij}$. So, i, j arbitrary $\Rightarrow \left(A^T\right)^T = A$.

Theorem 3.4(b): $\left[(A+B)^T\right]_{ij} = [A+B]_{ji} = [A]_{ji} + [B]_{ji} = \left[A^T\right]_{ij} + \left[B^T\right]_{ij}$.
i, j arbitrary $\Rightarrow (A+B)^T = A^T + B^T$.

Theorem 3.4(c): $\left[(kA)^T\right]_{ij} = [kA]_{ji} = k\,[A]_{ji} = k\left[A^T\right]_{ij}$.
i, j arbitrary $\Rightarrow (kA)^T = k\left(A^T\right)$.

33. We need to show $(A_1A_2\cdots A_n)^T = A_n^T\cdots A_2^TA_1^T$ for $n \geq 1$.
We will prove this using *induction.*
See Appendix B for discussion and examples of *Mathematical Induction.*

1: $(A_1)^T = A_1^T$. This is obvious, so there is nothing to show.

k: $(A_1A_2\cdots A_k)^T = A_k^T\cdots A_2^TA_1^T$.
This is the induction hypothesis, so there is nothing to show.

$k+1$: $(A_1 + A_2\cdots A_kA_{k+1})^T = A_{k+1}^TA_k^T\cdots A_2^TA_1^T$
This is the statement we must prove using the induction hypothesis.

$$(A_1A_2\cdots A_kA_{k+1})^T = ((A_1A_2\cdots A_k)\,A_{k+1})^T = A_{k+1}^T\,(A_1A_2\cdots A_k)^T \quad [\textit{by Thm } 3.4d]$$
$$= A_{k+1}^T\left(A_k^T\cdots A_2^TA_1^T\right) \quad [\textit{by induction}]$$

We have shown the pattern holds for $k+1$. What does that mean?
We have proven (by induction) that $(A_1A_2\cdots A_n)^T = A_n^T\cdots A_2^TA_1^T$.

37. For each matrix, we will simply check to see if $A^T = -A$ is satisfied.

(a) Since $A^T = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \neq -\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} = -A$, A is *not* skew-symmetric.

(b) Since $A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A$, A is skew-symmetric.

44. Let A and B be $n \times n$ matrices, and let k be a scalar. Then

(i) $\operatorname{tr}(A+B) = (a_{11}+b_{11}) + (a_{22}+b_{22}) + \cdots + (a_{nn}+b_{nn})$
$= (a_{11}+a_{22}+\cdots+a_{nn}) + (b_{11}+b_{22}+\cdots+b_{nn}) = tr\,(A) + tr\,(B)$.

(optional) (ii) $\operatorname{tr}(kA) = ka_{11} + ka_{22} + \cdots + ka_{nn} = k\,(a_{11}+a_{22}+\cdots+a_{nn}) = k\operatorname{tr}(A)$.

45. Let A and B be $n \times n$ matrices. Then

$$\begin{aligned}\operatorname{tr}(AB) &= (a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}) + (a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2}) + \\ &\qquad \cdots + (a_{n1}b_{1n} + a_{n2}b_{2n} + \cdots + a_{nn}b_{nn}) \\ &= (b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1n}a_{n1}) + (b_{21}a_{12} + b_{22}a_{22} + \cdots + b_{2n}a_{n2}) + \\ &\qquad \cdots + (b_{n1}a_{1n} + b_{n2}a_{2n} + \cdots + b_{nn}a_{nn}) \\ &= \operatorname{tr}(BA)\end{aligned}$$

(optional)

46. Let A be an $n \times n$ matrix. Then
$\operatorname{tr}\left(AA^T\right) = \left(a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2\right) + \left(a_{21}^2 + a_{22}^2 + \cdots + a_{2n}^2\right) + \cdots + \left(a_{n1}^2 + a_{n2}^2 + \cdots + a_{nn}^2\right)$,
that is, the sum of the squares of the entries of A.

3.3.

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14. To prove X is the inverse of A, all we have to show is $AX = I$.
Theorem 3.9b. asserts $(cA)^{-1} = \frac{1}{c}A^{-1}$, so all we need to show is $(cA)\left(\frac{1}{c}A^{-1}\right) = I$.

$$(cA)\left(\tfrac{1}{c}A^{-1}\right) = \left(\tfrac{1}{c}c\right)\left(AA^{-1}\right) = AA^{-1} = I$$

15. To prove X is the inverse of A, all we have to show is $AX = I$.
Theorem 3.9d. asserts $(A^T)^{-1} = (A^{-1})^T$, so all we need to show is $(A^T)(A^{-1})^T = I$.

$$(A^T)(A^{-1})^T \overset{\substack{\text{by Thm 3.4d} \\ B^TA^T=(AB)^T}}{=} (A^{-1}A)^T = I^T \overset{\substack{\text{This is obvious.} \\ \text{Why?}}}{=} I$$

16 Take $B = I_n$

$\Rightarrow BI_n = I_nB = I_nI_n = I_n$

$\Rightarrow I_n$ is invertible and $I_n^{-1} = I_n$ $\blacksquare$