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2. As is Example 3.24, we begin by computing the determinant of A , $\det A$. Why?

Since $A = \begin{bmatrix} 4 & -2 \\ 2 & 0 \end{bmatrix}$, $\det A = 4(0) - (-2)(2) = 4$, A is invertible.

So, by Theorem 3.8, $A^{-1} = \frac{1}{4} \begin{bmatrix} 0 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$.

Check: $AA^{-1} = \begin{bmatrix} 4 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ or $A^{-1}A = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 2 & 0 \end{bmatrix} = I$.

3. As is Example 3.24, we begin by computing the determinant of A , $\det A$. Why?

Since $A = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$, $\det A = 3(8) - 4(6) = 0$, A is not invertible.

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12. As in Example 3.25, we use the inverse of the coefficient matrix A to solve the system.

$$\text{Since } A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \text{ we have } A^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

$$\text{Therefore, since } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ we have } \mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\text{Check: } A\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{b}.$$

13. As in Example 3.25, we use the inverse of the coefficient matrix A to solve the systems.

$$\text{Since } A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}, \text{ we have } A^{-1} = \frac{1}{2} \begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix}.$$

$$\text{Also, note that we are given } \mathbf{b}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \text{ and } \mathbf{b}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

(a) Since $A\mathbf{x}_i = \mathbf{b}_i$, the solution in each case is $\mathbf{x}_i = A^{-1}\mathbf{b}_i$. So we have:

$$\begin{aligned} \mathbf{x}_1 = A^{-1}\mathbf{b}_1 &= \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -\frac{1}{2} \end{bmatrix} \\ \mathbf{x}_2 = A^{-1}\mathbf{b}_2 &= \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix} \\ \mathbf{x}_3 = A^{-1}\mathbf{b}_3 &= \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \end{aligned}$$

(b) We form the augmented matrix $[A \mid \mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3]$ and row reduce to solve.

$$\left[\begin{array}{cc|cc} 1 & 2 & 3 & -1 & 2 \\ 2 & 6 & 5 & 2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 4 & -5 & 6 \\ 0 & 1 & -\frac{1}{2} & 2 & -2 \end{array} \right]$$

(c) The A^{-1} method requires 7 multiplications, while row reduction requires only 6.

17. In (a) we will give an example, and in (b) we will prove $(AB)^{-1} = A^{-1}B^{-1} \Leftrightarrow AB = BA$.

(a) We choose A, B , then compute $(AB)^{-1}, A^{-1}B^{-1}$ to show $(AB)^{-1} \neq A^{-1}B^{-1}$.

Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$. Then

$$(AB)^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \right)^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix},$$

$$A^{-1}B^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

So, clearly $(AB)^{-1} \neq A^{-1}B^{-1}$.

$$22. (A^{-1}X)^{-1} = A(B^{-2}A)^{-1} \Rightarrow X^{-1}(A^{-1})^{-1} = A(A^{-1}(B^{-2})^{-1}) \Rightarrow X^{-1}A = B^2$$

$$\text{So } X = (B^2A^{-1})^{-1} = (A^{-1})^{-1}(B^2)^{-1} = AB^{-2}.$$

$$24. E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$36. \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$38. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.$$

40. As in Example 3.29, we attempt to express A as a product of elementary matrices. Why? To compute A^{-1} and to factor A , that is to see how A is created by elementary row operations.

As we row reduce A , we use the elementary operations involved to create E_i and E_i^{-1} .

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{So } E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \text{ and } E_4 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

$$\text{And } E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and } E_4^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Since $E_4 E_3 E_2 E_1 A = I$, we should have $A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$.

$$\text{Check: } A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

This answer is not unique. For example, we could have row reduced A as follows.

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{So } E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and } E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

$$\text{And } E_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and } E_4^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Verify that $A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$ in this case as well.

41. Suppose $AB = I$. Then, consider: $Bx = 0$. Left-multiplying by $A \Rightarrow ABx = A0$. This implies that $Ix = x = 0$. Thus $Bx = 0$ has the unique solution $x = 0 \Rightarrow$ The equivalence of (c) and (a) in the Fundamental Theorem (3.12) $\Rightarrow B$ is invertible. If we now right-multiply both sides of $AB = I$ by B^{-1} , we obtain

$$ABB^{-1} = IB^{-1} \Leftrightarrow AI = B^{-1} \Leftrightarrow A = B^{-1} \Leftrightarrow A^{-1} = B.$$

Thus the inverse of A exists and $A^{-1} = B$.

54. As in Example 3.30, we adjoin the identity matrix to A then row reduce $[A|I]$ to $[I|A^{-1}]$.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} \right] = [I|A^{-1}]$$

$$57. A^{-1} = \begin{bmatrix} -11 & -2 & 5 & -4 \\ 4 & 1 & -2 & 2 \\ 5 & 1 & -2 & 2 \\ 9 & 2 & -4 & 3 \end{bmatrix}$$

3.4

2. We solve the system by the two-step method outlined after Theorem 3.15:

$$Ax = b \stackrel{A=LU}{\Rightarrow} (LU)x = b \stackrel{Ux=y}{\Rightarrow} Ly = b$$

Since $A = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 0 & 4 \end{bmatrix} = LU$ and $b = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$, $Ly = b \Rightarrow$

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \Rightarrow \begin{matrix} y_1 = 0 \\ \frac{1}{2}y_1 + y_2 = 8 \end{matrix} \Rightarrow \begin{matrix} y_1 = 0 \\ y_2 = -\frac{1}{2}y_1 + 8 \end{matrix}$$

So, $y = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$.

Likewise, since $U = \begin{bmatrix} 4 & -2 \\ 0 & 4 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$, $Ux = y \Rightarrow$

$$\begin{bmatrix} 4 & -2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \Rightarrow \begin{matrix} 4x_1 - 2x_2 = 0 \\ 4x_2 = 8 \end{matrix} \Rightarrow \begin{matrix} x_1 = \frac{1}{2}x_2 \\ x_2 = 2 \end{matrix}$$

So, $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Check: $Ax = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix} = b$.

3. Since $A = LU$, where L is unit lower triangular and U is upper triangular, we have:

$$Ax = b \xrightarrow{A=LU} (LU)x = b \xrightarrow{(AB)C=A(BC)} L(Ux) = b \xrightarrow{Ux=y} Ly = b$$

So we solve the system by the two-step method outlined after Theorem 3.15:

- 1) Solve $Ly = b$ by *forward substitution* (see Exercises 25 and 26 in Section 2.1) and
- 2) Solve $Ux = y$ by *back substitution* (see Example 2.5 in Section 2.1)

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 4 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -\frac{5}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 0 & 4 & -6 \\ 0 & 0 & -\frac{7}{2} \end{bmatrix} = LU \text{ and } b = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \text{ so } Ly = b \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -\frac{5}{4} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= -3 & y_1 &= -3 \\ -y_1 + y_2 &= 1 & \Rightarrow y_2 &= y_1 + 1 \Rightarrow \\ 2y_1 - \frac{5}{4}y_2 + y_3 &= 0 & y_3 &= -2y_1 + \frac{5}{4}y_2 \end{aligned}$$

$$\begin{aligned} y_1 &= -3 \\ y_2 &= -3 + 1 = -2 \\ y_3 &= -2(-3) + \frac{5}{4}(-2) = \frac{7}{2} \end{aligned}$$

So, $y = \begin{bmatrix} -3 \\ -2 \\ \frac{7}{2} \end{bmatrix}$

Likewise, $U = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 4 & -6 \\ 0 & 0 & -\frac{7}{2} \end{bmatrix}$ and $y = \begin{bmatrix} -3 \\ -2 \\ \frac{7}{2} \end{bmatrix}$, so $Ux = y \Rightarrow$

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & 4 & -6 \\ 0 & 0 & -\frac{7}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ \frac{7}{2} \end{bmatrix} \Rightarrow \begin{aligned} 2x_1 + x_2 - 2x_3 &= -3 & x_1 &= -\frac{1}{2}x_2 + x_3 - \frac{3}{2} \\ 4x_2 - 6x_3 &= -2 & \Rightarrow x_2 &= \frac{3}{2}x_3 - \frac{1}{2} \\ -\frac{7}{2}x_3 &= \frac{7}{2} & x_3 &= -1 \end{aligned}$$

$$\begin{aligned} x_1 &= -\frac{1}{2}(-2) - 1 - \frac{3}{2} = -\frac{3}{2} \\ x_2 &= \frac{3}{2}(-1) - \frac{1}{2} = -2 \\ x_3 &= -1 \end{aligned}$$

So, $x = \begin{bmatrix} -\frac{3}{2} \\ -2 \\ -1 \end{bmatrix}$

Check: $Ax = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 4 & -3 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = b.$

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8. Following the multiplier method of Example 3.35, we find the LU factorization of A:

$$A = \begin{bmatrix} 2 & -4 \\ 3 & 1 \end{bmatrix} \xrightarrow{R'_2 = R_2 - \frac{3}{2}R_1} \begin{bmatrix} 2 & -4 \\ 0 & 7 \end{bmatrix} = U \Rightarrow l_{21} = \frac{3}{2} \Rightarrow L = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}$$

$$\text{Check: } LU = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 3 & 1 \end{bmatrix} = A$$

10. Following the multiplier method of Example 3.35, we find the LU factorization of A:

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 4 & 0 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R'_2 = R_2 - 2R_1} \begin{bmatrix} 2 & 2 & -1 \\ 0 & -4 & 6 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{R'_3 = R_3 - \frac{3}{2}R_1} \begin{bmatrix} 2 & 2 & -1 \\ 0 & -4 & 6 \\ 0 & 1 & \frac{11}{2} \end{bmatrix} \xrightarrow{R''_3 = R'_3 + \frac{1}{4}R'_2} \begin{bmatrix} 2 & 2 & -1 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix} = U \Rightarrow$$

$$l_{21} = 2 \\ l_{31} = \frac{3}{2} \quad l_{32} = -\frac{1}{4} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{3}{2} & -\frac{1}{4} & 1 \end{bmatrix}$$

$$\text{Check: } LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{3}{2} & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 \\ 4 & 0 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

6. 1

1. Let $V = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} \right\}$. We will show V satisfies all 10 axioms required of a vector space.

1. $\begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} y \\ y \end{bmatrix} \in V \Rightarrow \begin{bmatrix} x+y \\ x+y \end{bmatrix} \in V$.
2. $\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} y \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix} + \begin{bmatrix} x \\ x \end{bmatrix}$.
3. $\left(\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} y \\ y \end{bmatrix} \right) + \begin{bmatrix} z \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ x+y+z \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} + \left(\begin{bmatrix} y \\ y \end{bmatrix} + \begin{bmatrix} z \\ z \end{bmatrix} \right)$.
4. $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$ and $\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix}$.
5. $\begin{bmatrix} x \\ x \end{bmatrix} \in V \Rightarrow \begin{bmatrix} -x \\ -x \end{bmatrix} \in V$ and $\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} -x \\ -x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
6. $\begin{bmatrix} x \\ x \end{bmatrix} \in V \Rightarrow \begin{bmatrix} cx \\ cx \end{bmatrix} \in V$.
7. $c \left(\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} y \\ y \end{bmatrix} \right) = \begin{bmatrix} cx+cy \\ cx+cy \end{bmatrix} = c \begin{bmatrix} x \\ x \end{bmatrix} + c \begin{bmatrix} y \\ y \end{bmatrix}$.
8. $(c+d) \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} cx+dx \\ cx+dx \end{bmatrix} = c \begin{bmatrix} x \\ x \end{bmatrix} + d \begin{bmatrix} x \\ x \end{bmatrix}$.
9. $c \left(d \begin{bmatrix} x \\ x \end{bmatrix} \right) = \begin{bmatrix} cdx \\ cdx \end{bmatrix} = (cd) \begin{bmatrix} x \\ x \end{bmatrix}$.
10. $1 \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} 1x \\ 1x \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix}$.

2. Since $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \geq 0 \right\}$ fails to satisfy Axioms 5 and 6, V is not a vector space.

5. $\begin{bmatrix} x \\ y \end{bmatrix} \in V, c < 0 \Rightarrow c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix} \notin V$ because $cx, cy \leq 0$.
6. $\begin{bmatrix} x \\ y \end{bmatrix} \in V \Rightarrow \begin{bmatrix} -x \\ -y \end{bmatrix} \notin V$ because $-x, -y \leq 0$.

5. $\begin{bmatrix} cx \\ y \end{bmatrix} + \begin{bmatrix} dx \\ y \end{bmatrix} \neq \begin{bmatrix} (c+d)x \\ y \end{bmatrix} \Rightarrow$ axiom 8 fails $\Rightarrow V$ is not a vector space.

7. All axioms apply $\Rightarrow V$ is a vector space.

- | | | |
|--------------------------------------|--------------------------|--------------------------------|
| 1. $x > 0, y > 0 \Rightarrow xy > 0$ | 2. $xy = yx$ | 3. $(xy)z = x(yz)$ |
| 4. $x1 = x$ (so $1 = 0!$) | 5. $x^{\frac{1}{x}} = 1$ | 6. $x > 0 \Rightarrow x^c > 0$ |
| 7. $(xy)^c = x^c y^c$ | 8. $x^{(c+d)} = x^c x^d$ | 9. $(x^d)^c = x^{dc}$ |
| 10. $x^1 = x$ | | |

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9. All axioms apply $\Rightarrow V$ is a vector space. We will show only the key axioms of 1 and 6.

$$1. \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ 0 & c+c' \end{bmatrix} \quad 6. d \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} da & db \\ 0 & dc \end{bmatrix}$$

12. Need only show axioms 8, 9 and 10 are satisfied.

$$8. c(p(x) + q(x)) = c(a_0 + a_1x + a_2x^2) + c(b_0 + b_1x + b_2x^2) = cp(x) + cq(x).$$

$$9. c(dp(x)) = c(da_0 + da_1x + da_2x^2) = cda_0 + cda_1x + cda_2x^2 = (cd)p(x).$$

$$10. 1p(x) = 1(a_0 + a_1x + a_2x^2) = a_0 + a_1x + a_2x^2 = p(x).$$

14. $c \begin{bmatrix} z \\ \bar{z} \end{bmatrix} = \begin{bmatrix} cz \\ c\bar{z} \end{bmatrix} \neq \begin{bmatrix} z \\ \bar{z} \end{bmatrix} \Rightarrow$ axiom 6 fails $\Rightarrow V$ is not a complex vector space.

23. Will show $u + (-1)u = 0 \Rightarrow (-1)u = -u : u + (-1)u = 1u + (-1)u = (1 + (-1))u = 0u = 0$.

24. $\begin{bmatrix} a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} b \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ 0 \\ a+b \end{bmatrix}, c \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} = \begin{bmatrix} ca \\ 0 \\ ca \end{bmatrix} \Rightarrow W$ is a subspace.

27. $\begin{bmatrix} a \\ b \\ |a| \end{bmatrix} + \begin{bmatrix} a' \\ b' \\ |a'| \end{bmatrix} = \begin{bmatrix} a+a' \\ b+b' \\ |a|+|a'| \end{bmatrix} \neq \begin{bmatrix} a+a' \\ b+b' \\ |a+a'| \end{bmatrix} \Rightarrow W$ is not a subspace.

(optional)

29. $a, b \neq 0, \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} -a & a \\ b & -b \end{bmatrix} \in W$ but $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} -a & a \\ b & -b \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \notin W \Rightarrow$ no.

34. We will show that W satisfies axioms 1 and 6 $\Rightarrow W$ is a subspace.

$$1. v + v' = (bx + cx^2) + (b'x + c'x^2) = (b + b')x + (c + c')x^2 \Rightarrow v + v' \in W.$$

$$6. dv = d(bx + cx^2) = dbx + dcx^2 \Rightarrow dv \in W.$$

37. $v = x^3, w = -x^3 \in W \Rightarrow v + w = x^3 + (-x^3) = 0 \notin W$ (because degree = 0) \Rightarrow no.