HW 7 Solutions

math 321

DP 6.1

35. We will show that W satisfies axioms 1 and 6 $\Rightarrow W$ is a subspace.

1. Let $\mathbf{v} = a + b + cx^2$, $\mathbf{v}' = a' + b' + c'x^2 \in W$, that is $a + b + c = 0$ and $a' + b' + c' = 0$.
   Then $(a + a') + (b + b') + (c + c') = 0 \Rightarrow \mathbf{v} + \mathbf{v}' \in W$.
6. Let $\mathbf{v} = a + b + cx^2 \in W$, that is $a + b + c = 0$.
   Then $c(a + b + c) = 0 \Rightarrow c\mathbf{v} \in W$.

DP 3.5

2. We should suspect that set S defined by $x \geq 0$ and $y \geq 0$ is not a subspace. Why?
Because multiplication by scalars (e.g., negative numbers) remove values from the set S.

How do we prove that S is *not* a subspace?
We provide a counterexample to show that one of the required properties does *not* hold

So, note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in S, but $-1\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ is not.

Property (3) ($\mathbf{u}$ in S implies $c\mathbf{u}$ in S) fails so S is not a subspace of $\mathbb{R}^2$.

7. We should recognize right away that set S defined by $x - y + z = 1$ is *not* a subspace. Why?
Because $x - y + z = 1$ is a plane that does *not* pass through the origin. So we conclude:

Property (1) ($\mathbf{0}$ is in S) fails so S is not a subspace of $\mathbb{R}^3$.

Q: How can we tell that $x - y + z = 1$? is a plane that does *not* pass through the origin?

A: The equation of a plane through the origin must be of the form $ax + by + cz = 0$. Why?
Because the zero vector must satisfy the equation. So, we must have: $a0 + b0 + c0 = 0$.

11. As in Example 3.41, we will determine whether $\mathbf{b}$ is in col(A) and $\mathbf{w}$ is in row(A).

col(A): To say $\mathbf{b}$ is in col(A) means $\mathbf{b}$ is a linear combination of the columns of A. So:

$\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is in the column space of $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ if the system $A\mathbf{x} = \mathbf{b}$ is consistent.

We row reduce the augmented matrix: $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 1 & 1 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right] \Rightarrow$

The system is consistent. So, $\mathbf{b} \in$ col(A). In particular, $\mathbf{b} = 3\mathbf{a}_1 - \mathbf{a}_2$.

row(A): To say $\mathbf{w}$ is in row(A) means $\mathbf{w}$ is a linear combination of the rows of A. So:

$\mathbf{w} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \in$ row(A) if $\left[\begin{array}{c} A \\ \hline \mathbf{w} \end{array}\right] \longrightarrow \left[\begin{array}{c} A \\ \hline \mathbf{0} \end{array}\right]$

by elementary row operations *excluding* row interchanges involving the *last* row.

So, we have $\left[\begin{array}{c} A \\ \hline \mathbf{w} \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & -1 \\ 1 & 1 & 1 \\ \hline -1 & 1 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-R_1} \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 2 \\ \hline 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 2 \\ \hline 0 & 0 & -2 \end{array}\right] \Rightarrow$

We cannot make the last row all zeroes $\Rightarrow \mathbf{w} \notin$ row(A).

12. As in Example 3.41, we will determine whether $\mathbf{b}$ is in col(A) and $\mathbf{w}$ is in row(A).

col(A): To say $\mathbf{b}$ is in col(A) means $\mathbf{b}$ is a linear combination of the columns of A. So:

$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is in the column space of $A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix}$ if $A\mathbf{x} = \mathbf{b}$ is consistent.

We row reduce the augmented matrix: $\left[\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & -1 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$

The system is consistent. So, $\mathbf{b} \in$ col(A). In particular, $\mathbf{b} = \frac{1}{2}\mathbf{a}_1 + \frac{1}{2}\mathbf{a}_2(A)$.

row(A): To say $\mathbf{w}$ is in row(A) means $\mathbf{w}$ is a linear combination of the rows of A. So:

$\mathbf{w} = \begin{bmatrix} 2 & 4 & -5 \end{bmatrix} \in$ row(A) if $\left[\begin{array}{c} A \\ \hline \mathbf{w} \end{array}\right] \longrightarrow \left[\begin{array}{c} A \\ \hline \mathbf{0} \end{array}\right]$

by elementary row operations *excluding* row interchanges involving the *last* row.

So, $\left[\begin{array}{c} A \\ \hline \mathbf{w} \end{array}\right] = \left[\begin{array}{ccc} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \\ \hline 2 & 4 & -5 \end{array}\right] \xrightarrow[R_4-2R_1]{R_3-R_1} \left[\begin{array}{ccc} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \\ \hline 0 & 2 & 1 \end{array}\right] \xrightarrow[R_4-R_2]{R_3+R_2} \left[\begin{array}{ccc} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \Rightarrow$

So, $\mathbf{w}$ is a linear combination of the rows of $A \Rightarrow \mathbf{w} \in$ row(A).

16. Since $A\mathbf{v} = \mathbf{0}$ implies $\mathbf{v}$ is in null(A), we simply multiply $A\mathbf{v}$ to check:

$$A\mathbf{v} = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} \Rightarrow \mathbf{v} \in \text{null}(A).$$

DP 6.2

1. Let $V_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, V_2 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, V_3 = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$.

Then, $aV_1 + bV_2 + cV_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a+b+c=0 & a-b=0 \\ b+3c=0 & -a+2c=0 \end{bmatrix} \Rightarrow a = b = c = 0 \Rightarrow$ V_1, V_2, V_3 linearly independent by definition.

5. Let $p(x) = x$, $q(x) = 1 + x$, then $a\,p(x) + b\,q(x) = a(x) + b(1+x) = b + (a+b)x = 0 \Rightarrow$ $b = 0 \;\; a + b = 0 \;\Rightarrow a = b = 0 \Rightarrow p(x), q(x)$ linearly independent.

6. Let $p(x) = 1 + x$, $q(x) = 1 + x^2$, $r(x) = 1 - x + x^2$.
Then $a\,p(x) + b\,q(x) + c\,r(x) = (a+b+c) + (a-c)x + (b+c)x^2 = 0 \Rightarrow$
$a+b+c = 0,\; a-c = 0,\; b+c = 0 \;\Rightarrow a = b = c = 0 \Rightarrow$ linearly independent.

(optional)

17. (a) $a(\mathbf{u}+\mathbf{v}) + b(\mathbf{v}+\mathbf{w}) + c(\mathbf{u}+\mathbf{w}) = (a+c)\mathbf{u} + (a+b)\mathbf{v} + (b+c)\mathbf{w} = \mathbf{0} = 0 \Rightarrow$
$a+c = 0,\, a+b = 0,\, b+c = 0 \Rightarrow a = b = c = 0 \Rightarrow$
$\mathbf{u}+\mathbf{v},\; \mathbf{v}+\mathbf{w},\; \mathbf{u}+\mathbf{w}$ are linearly independent.

(b) $a(\mathbf{u}-\mathbf{v}) + b(\mathbf{v}-\mathbf{w}) + c(\mathbf{u}-\mathbf{w}) = (a+c)\mathbf{u} + (-a+b)\mathbf{v} + (-b-c)\mathbf{w} = \mathbf{0} = 0 \Rightarrow$
$a+c = 0,\; -a+b = 0,\; -b-c = 0 \Rightarrow a = b = 1,\; c = -1$ is a solution
$\Rightarrow \mathbf{u}-\mathbf{v},\; \mathbf{v}-\mathbf{w},\; \mathbf{u}-\mathbf{w}$ are linearly dependent.
Note, a was free so there are infinitely many solutions.
Eg.: Let $\mathbf{u} = 1$, $\mathbf{v} = x$, $\mathbf{w} = x^2$, then $\mathbf{u}-\mathbf{v} = 1 - x$, $\mathbf{v}-\mathbf{w} = x - x^2$, $\mathbf{u}-\mathbf{w} = 1 - x^2$.
Since $1 - x^2 = 1(1-x) + 1(x - x^2)$, $\mathbf{u}-\mathbf{v},\; \mathbf{v}-\mathbf{w},\; \mathbf{u}-\mathbf{w}$ are clearly linearly dependent.

Dr 3.5

17. We find bases for row(A), col(A), and null(A) as in Examples 3.45, 3.47, and 3.48 respectively.

row(A): A basis for row(A) must span the rows of A and be linearly independent.
Given $A \longrightarrow R$, Theorem 3.20 asserts that the rows of R span the rows of A. Why?
Because the rows of A are linear combinations of the rows of R (and vice-versa).
Finally, we simply observe that the nonzero rows of R are linearly independent.

$$\text{Since } A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = R,$$

we conclude that $\{[\,1\ \ 0\ \ -1\,], [\,0\ \ 1\ \ 2\,]\}$ is a basis for row(A).

We should also note that provided $A \longrightarrow R$ uses no row interchanges,
the corresponding rows in A are also linearly independent.
Whence, it is obvious that those rows form a basis for row(A).

col(A): A basis for col(A) must span the columns of A and be linearly independent.
When $A \longrightarrow R$, the columns with leading 1s in R are linearly independent.
As shown in Example 3.47, the corresponding columns in A are also linearly independent.
Whence, it is obvious that those columns form a basis for col(A).

$$\text{Since } A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = R,$$

we conclude that $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for col(A).

null(A): Since $A\mathbf{v} = \mathbf{0}$ implies $\mathbf{v}$ is in null(A), we solve $[\,A \,|\, \mathbf{0}\,] \longrightarrow [\,R \,|\, \mathbf{0}\,]$ to find the conditions:

$$[\,R \,|\, \mathbf{0}\,] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \\ x_3 \ \mathit{free} \end{array} \Rightarrow \begin{array}{rcr} x_1 & = & 1s \\ x_2 & = & -2s \\ x_3 & = & 1s \end{array}$$

Since t is arbitrary, null(A) $= \operatorname{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right)$. So, $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for null(A).

18. We find bases for row(A), col(A), and null(A) as in Examples 3.45, 3.47, and 3.48 respectively.

row(A): A basis for row(A) must span the rows of A and be linearly independent.
Given $A \longrightarrow U$, Theorem 3.20 asserts that the rows of U span the rows of A. Why?
Because the rows of A are linear combinations of the rows of U (and vice-versa).
Finally, we simply observe that the nonzero rows of U are linearly independent.

$$\text{Since } A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix} \xrightarrow{R_3 - R_1 + R_2} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = U,$$

we conclude that $\left\{ \begin{bmatrix} 1 & 1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \end{bmatrix} \right\}$ is a basis for row(A).

Q: In $A \longrightarrow U$, why is it sufficient to reduce A only to row echelon form U?

A: As the remark following Example 3.46 explains and then demonstrates by example, the nonzero rows of U are linearly independent. That is all that is required. Why?

We should also note that provided $A \longrightarrow U$ uses no row interchanges,
the corresponding rows in A are also linearly independent.
Whence, it is obvious that those rows form a basis for row(A).

col(A): A basis for col(A) must span the columns of A and be linearly independent.
When $A \longrightarrow U$, the columns with leading entries in U are linearly independent.
As in Example 3.47, the corresponding columns in A are also linearly independent.
Whence, it is obvious that those columns form a basis for col(A).

$$\text{Since } A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = U,$$

we conclude that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ is a basis for col(A).

null(A): Since $A\mathbf{v} = \mathbf{0}$ implies $\mathbf{v}$ is in null(A), we solve $\left[\, A \,|\, \mathbf{0} \,\right] \longrightarrow \left[\, U' \,|\, \mathbf{0} \,\right]$ to find the conditions.
We row reduce U one more step to U' make it easier to find the conditions:

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix} \xrightarrow{R_3 - R_1 + R_2} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + 3R_2} \begin{bmatrix} 1 & 7 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = U'$$

$$\left[\, U' \,|\, \mathbf{0} \,\right] = \left[\begin{array}{ccc|c} 1 & 7 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + 7x_2 = 0 \\ 2x_2 + x_3 = 0 \\ x_2 \textit{ free} \end{array} \Rightarrow \begin{array}{rcl} x_1 & = & -7s \\ x_2 & = & s \\ x_3 & = & -2s \end{array}$$

Since s is arbitrary, null$(A) = \text{span}\left(\begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix} \right)$. So, $\left\{ \begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix} \right\}$ is a basis for null(A).

19. We find bases for row(A), col(A), and null(A) as in Examples 3.45, 3.47, and 3.48 respectively.

row(A): A basis for row(A) must span the rows of A and be linearly independent.
Given $A \longrightarrow R$, Theorem 3.20 asserts that the rows of R span the rows of A. Why?
Because the rows of A are linear combinations of the rows of R (and vice-versa).
Finally, we simply observe that the nonzero rows of R are linearly independent.

$$\text{Since } A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = R,$$

we conclude that $\left\{ \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right\}$ is a basis for row(A).
We should also note that provided $A \longrightarrow R$ uses no row interchanges,
the corresponding rows in A are also linearly independent.
Whence, it is obvious that those rows form a basis for row(A).

col(A): A basis for col(A) must span the columns of A and be linearly independent.
When $A \longrightarrow R$, the columns with leading 1s in R are linearly independent.
As shown in Example 3.47, the corresponding columns in A are also linearly independent.
Whence, it is obvious that those columns form a basis for col(A).

$$\text{Since } A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = R,$$

$$\text{we conclude that } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\} \text{ is a basis for col}(A).$$

null(A): Since $A\mathbf{v} = \mathbf{0}$ implies $\mathbf{v}$ is in null(A), we solve $\left[\, A \,|\, \mathbf{0} \,\right] \longrightarrow \left[\, R \,|\, \mathbf{0} \,\right]$ to find the conditions.

$$\left[\, R \,|\, \mathbf{0} \,\right] = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{array}{rcl} x_1 + x_3 & = & 0 \\ x_2 - x_3 & = & 0 \\ x_3 \; free & & \\ x_4 & = & 0 \end{array} \Rightarrow \begin{array}{rcr} x_1 & = & -s \\ x_2 & = & s \\ x_3 & = & s \\ x_4 & = & 0 \end{array}$$

$$\text{Since } s \text{ is arbitrary, null}(A) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right). \text{ So, } \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis for null}(A).$$