HW 8 Solutions math 327

3.5

21. We find bases for row(A) and col(A) following Examples 3.45 and 3.47 respectively.

row(A): A basis for col(A) must span the columns of A and be linearly independent.
Clearly, the linearly independent *columns* of A^T do just that.
When $A^T \longrightarrow R$, the columns with leading 1s in R are linearly independent.
As in Example 3.47, the corresponding columns in A^T are also linearly independent.
Whence, it is obvious that the *transposes* of those columns form a basis for row(A).

$$\text{Since } A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = R$$

we conclude that $\left\{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\right\}$ is a basis for row(A).

col(A): A basis for col(A) must span the columns of A and be linearly independent.
When $A^T \longrightarrow R$, the linearly independent *rows* (the nonzero rows) of R do just that.
Whence, it is obvious that the *transposes* of those rows form a basis for col(A).

$$\text{Since } A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = R$$

we conclude that $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis for col(A).

We should also note that provided $A^T \longrightarrow R$ uses no row interchanges,
the corresponding rows in A^T are also linearly independent.
Whence, it is obvious that the *transposes* of those rows form a basis for col(A).

22. We find bases for row(A) and col(A) following Examples 3.45 and 3.47 respectively.

row(A): A basis for col(A) must span the columns of A and be linearly independent.
Clearly, the linearly independent *columns* of A^T do just that.
When $A^T \longrightarrow R$, the columns with leading 1s in R are linearly independent.
As in Example 3.47, the corresponding columns in A^T are also linearly independent.
Whence, it is obvious that the *transposes* of those columns form a basis for row(A).

$$\text{Since } A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -3 & 1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = R,$$

we conclude that $\left\{\begin{bmatrix} 1 & 1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}\right\}$ is a basis for row(A).

col(A): A basis for col(A) must span the columns of A and be linearly independent.
When $A^T \longrightarrow R$, the linearly independent *rows* (the nonzero rows) of R do just that.
Whence, it is obvious that the *transposes* of those rows form a basis for col(A).

$$\text{Since } A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -3 & 1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = R,$$

we conclude that $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$ is a basis for col(A).

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25. Our answers to Exercises 17 and 21 appear different because we used different methods. Let's compare the methods and answers for row(A) and col(A) from each of the exercises.

row(A): In Exercise 17, we found the basis for row(A) as follows:
Given $A \longrightarrow R$, the linearly independent (nonzero) rows of R span the rows of A. Thus:

$$\text{Since } A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = R,$$

we conclude that $\{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}\}$ is a basis for row(A).

In Exercise 21, on the other hand, we found the basis for row(A) as follows:
When $A^T \longrightarrow R$, the *transposes* of the *columns* in A^T corresponding to the columns with leading 1s in R form a basis for row(A). Thus:

$$\text{Since } A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = R$$

we conclude that $\{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\}$ is a basis for row(A).

So, first we used rows of R as our basis, then we used rows of A.
Do the rows of A corresponding to the nonzero rows of R form a basis?
Yes. Since $A \longrightarrow R$ uses no row interchanges, those rows are linearly independent.
We prove this explicitly by showing the spans of these two sets are equal.
By Exercise 21 of Section 2.3, we need only observe:

$$-\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}.$$

Why is this enough? The basis vectors in each set are linear combinations of each other.

col(A): In Exercise 17, we found the basis for col(A) as follows:
When $A \longrightarrow R$, the columns in A corresponding to the columns with leading 1s in R form a basis for col(A). Thus:

$$\text{Since } A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = R,$$

we conclude that $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis for col(A).

In Exercise 21, on the other hand, we found the basis for col(A) as follows:
Given $A^T \longrightarrow R$, the *transposes* of the linearly independent (nonzero) *rows* of R span the columns of A. Thus:

$$\text{Since } A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = R$$

we conclude that $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis for col(A).

So, first we used columns of A as our basis, then we used *transposes* of the rows of R.
Notice, however, those rows correspond to the columns of A found in Exercise 17.

For example, $\begin{bmatrix} 1 & 0 \end{bmatrix}$ of R corresponds to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ of A.

Explicitly, it is obvious that the span of both these sets is $\mathbb{R}^2$. Why?
Since both sets contain two vectors and the dimension of $\mathbb{R}^2$ is obviously 2.

27. As in Example 3.46, given $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ we form matrix B and row reduce:

Since $B = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_3+R_1+R_2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

we conclude that $\{\mathbf{u}, \mathbf{v}\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for span(S).

Note: We rearrange and transpose the vectors of S to simplify the row reduction.
As noted in the remark following Example 3.46, we need only reduce to row echelon form.
Then, we find the basis by identifying linearly independent vectors in the original set.

Q: Does $\{\mathbf{u}, \mathbf{w}\}$ also form a basis for S? What about $\{\mathbf{v}, \mathbf{w}\}$?
A: Yes, since no 2 vectors are multiples of each other. Why is that enough?

28. Before we begin, we should note that at most 3 of these vectors can be linearly independent.
Furthermore, we should let this insight inform our formation of matrix B. How?
As in Example 3.46, given $S = \{\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w}\}$ we form matrix B and row reduce:

Since $B = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{w}^T \\ \mathbf{v}^T \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow{R_3-2R_1+3R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix} = U$,

we conclude that $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} \right\}$ is a basis for span(S).

Q: Could we also conclude $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for S after only one step?
A: Yes, since the rows of B corresponding to the linearly independent (nonzero) rows of U are linearly independent. This is only true because we performed no row interchanges.

33. Let R be a matrix in echelon form. Then row(R) = span(the rows of R) by definition.
Nonzero rows of R are linearly independent (first entries are in different columns) $\Rightarrow$
Nonzero rows of R form a basis for row(R), by definition.

35. We use our work from Exercise 17 to determine rank(A) and nullity(A) below.

rank(A) = number of nonzero rows in R = number of vectors in a basis for row(A) or col(A)

Since $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = R$,
and $\{[\,1 \;\; 0 \; -1\,], [\,0 \;\; 1 \;\; 2\,]\}$ is a basis for row(A), we have rank(A) = 2.

nullity(A) = n − rank(A) = number of nonzero vectors in a basis for null(A)

From Exercise 17, $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for null(A), so nullity(A) = 3 − 2 = 1.

36. We use our work from Exercise 18 to determine rank(A) and nullity(A) below.

rank(A) = number of nonzero rows in U = number of vectors in a basis for col(A) or row(A)

Since $A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = U,$

and $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ is a basis for col(A), we have rank$(A) = 2$.

nullity$(A) = n -$ rank(A) = number of nonzero vectors in a basis for null(A)

From Exercise 18, $\left\{ \begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix} \right\}$ is a basis for null(A), so nullity$(A) = 3 - 2 = 1$.

39. If nullity$(A) > 0$, then the columns of A are linearly dependent.
Though we could prove this using theorems, it is instructive to prove it directly.

If nullity$(A) > 0$, then there exists a vector $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$.

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $\mathbf{x}^T = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$.

Since $\mathbf{x} \neq \mathbf{0}$ at least one $c_i \neq 0$. Then $A\mathbf{x} = \sum c_i \mathbf{a}_i = \mathbf{0}$ where at least one $c_i \neq 0$.
Therefore, the columns of A are linearly dependent.

So all we have to show is: If A is a 3×5 matrix, then nullity$(A) > 0$.

$$\text{nullity}(A) \overset{\substack{\text{Rank}\\\text{Thm}}}{=} n - \text{rank}(A) \overset{\substack{A\text{ is}\\3\times5}}{=} 5 - \text{rank}(A) \overset{\text{rank}(A)\leq 3}{\geq} 5 - 3 = 2 > 0$$

Q: If A is 3×5, why is it obvious that rank$(A) \leq 3$?
A: Recall, that the number of vectors in a basis for row(A) = dim(row(A)).
Now note the rows contain a basis for row(A), so dim(row(A)) $\leq$ number of rows.
So, rank(A) = dim(row(A)) $\leq$ the number of rows = 3.

41. Rank must satisfy the following two conditions simultaneously:
1) rank(A) = dim(row(A)) $\leq$ the number of rows
2) rank(A) = dim(col(A)) $\leq$ the number of columns

Therefore, rank must be less than or equal to the smaller of these two numbers.

Since A is 3×5, rank(A) can equal 0, 1, 2, or 3.
Therefore, since $n = 5$ and nullity$(A) = n -$ rank(A), we have:

nullity$(A) = 5 - 3 = 2$, $5 - 2 = 3$, $5 - 1 = 4$, or $5 - 0 = 5$

44. $A = \begin{bmatrix} a & 2 & -1 \\ 3 & 3 & -2 \\ -2 & -1 & a \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} -2 & -1 & a \\ 3 & 3 & -2 \\ a & 2 & -1 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 2 & a-2 \\ 3 & 3 & -2 \\ a & 2 & -1 \end{bmatrix} \longrightarrow$

$$\begin{bmatrix} 1 & 2 & a-2 \\ 0 & 1 & a-\frac{4}{3} \\ 0 & 2-2a & -(a-1)^2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & a-2 \\ 0 & 1 & a-\frac{4}{3} \\ 0 & 0 & (a-1)(a-\frac{5}{3}) \end{bmatrix} \Rightarrow$$

If $a = 1, \frac{5}{3}$, then $A \longrightarrow \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(A) = 2$. Otherwise, $\text{rank}(A) = 3$.

45. As in Example 3.52, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ form a basis for $\mathbb{R}^3 \Leftrightarrow$ When $A = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix}$, $\text{rank}(A) = 3$.

$$A = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since $\text{rank}(A) = 3$, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ form a basis for $\mathbb{R}^3$.

47. As in Example 3.52, $\{\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w}\}$ form a basis for $\mathbb{R}^4 \Leftrightarrow$ When $A = \begin{bmatrix} \mathbf{x}^T \\ \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix}$, $\text{rank}(A) = 4$.

$$A = \begin{bmatrix} \mathbf{x}^T \\ \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 + R_2 + R_4} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Since $\text{rank}(A) = 4$, $\{\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w}\}$ form a basis for $\mathbb{R}^4$.

55. We need to show if $\mathbf{v}$ is in $\text{row}(A)$ and $\mathbf{x}$ is in $\text{null}(A)$, then $\mathbf{v} \cdot \mathbf{x} = 0$.

First, we will show if $\mathbf{x}$ is in $\text{null}(A)$ and $\mathbf{A}_i$ is the ith row of A, then $\mathbf{A}_i \cdot \mathbf{x} = 0$.

Let $A = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_m \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Then we have: If $\mathbf{x}$ is in $\text{null}(A)$, then $A\mathbf{x} = \mathbf{0}$.

$$\text{So, } A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 \cdot \mathbf{x} \\ \mathbf{A}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{A}_m \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Therefore, if $\mathbf{x}$ is in $\text{null}(A)$, then $\mathbf{A}_i \cdot \mathbf{x} = 0$.

If $\mathbf{v}$ is in $\text{row}(A)$, then $\mathbf{v}$ is a linear combination of the rows of A so $\mathbf{v} = \sum c_i A_i$.

So, we have: $\mathbf{v} \cdot \mathbf{x} = (\sum c_i A_i) \cdot \mathbf{x} = \sum c_i (A_i \cdot \mathbf{x}) = \sum c_i 0 = 0$.

Q: What is the idea behind the proof of this exercise?

A: If a vector is orthogonal to a set of vectors,
it is orthogonal to all linear combinations of those vectors.

6

(optional)

57. We will prove part (a) using the idea we suggested by Exercise 29 of Section 3.1.

(a) Let $\{\mathbf{Ab}_k\}$ be a basis for col(AB) formed from the columns of AB.
Then rank(AB) = number of vectors in $\{\mathbf{Ab}_k\}$ = number of vectors in $\{\mathbf{b}_k\}$.
First we show: If $\{\mathbf{Ab}_k\}$ is linearly independent, then $\{\mathbf{b}_k\}$ is linearly independent.
That is, if $\sum c_k\mathbf{b}_k = \mathbf{0}$, we need to show all the $c_k = 0$.
$\sum c_k\mathbf{b}_k = \mathbf{0} \Rightarrow A(\sum c_k\mathbf{b}_k) = \mathbf{0} \Rightarrow \sum c_k(A\mathbf{b}_k) = \mathbf{0}$
Since $\{\mathbf{Ab}_k\}$ is linearly independent by assumption, all the $c_k = 0$ as required.
Now, since $\{\mathbf{b}_k\}$ is a linearly independent subspace of the columns of B, the number of vectors in $\{\mathbf{b}_k\} \leq$ rank(B).
Therefore, rank(AB) = number of vectors in $\{\mathbf{b}_k\} \leq$ rank(B).

(b) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then B has rank 2, but $AB = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ has rank 1.

(optional)

60. Since this is an if and only if statement, there are two statements to prove.

if: If rank$(A) = 1$ then col(A) = span$(\mathbf{u})$, so $A = \mathbf{u}\mathbf{v}^T$ where
$\mathbf{a}_i = c_i\mathbf{u}$ are the columns of A and $\mathbf{v}^T = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$.
If rank$(A) = 1$, a basis for col(A) has only one vector. That is, col(A) = span$(\mathbf{u})$.
Furthermore, every column of A must be a multiple of that vector.
That is $\mathbf{a}_i = c_i\mathbf{u}$ for every column of A and $A = \begin{bmatrix} c_1\mathbf{u} & \cdots & c_n\mathbf{u} \end{bmatrix}$.
So, if we let $\mathbf{v}^T = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$, then $A = \mathbf{u}\mathbf{v}^T$ as required.

only if: If $A = \mathbf{u}\mathbf{v}^T$ then col(A) = span$(\mathbf{u})$, so rank$(A) = 1$.
Why? Because rank(A) = the number of vectors in a basis for col(A).
Let $\mathbf{v}^T = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$, then $A = \mathbf{u}\mathbf{v}^T = \begin{bmatrix} c_1\mathbf{u} & \cdots & c_n\mathbf{u} \end{bmatrix}$.
So, $\mathbf{a}_i = c_i\mathbf{u}$ for every column of $A \Rightarrow$ col(A) = span$(\mathbf{u}) \Rightarrow$ rank$(A) = 1$.

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6.2

37. $\mathcal{B} = \{E_{11}, E_{12}, E_{22}\}$ is a basis $\Rightarrow \dim V = 3$.

38. Recall *skew* symmetric means $A^T = -A \Rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix} = -\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow$

$a = d = 0$, $c = -b$, b free $\Rightarrow \mathcal{B} = \left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$ is a basis $\Rightarrow \dim V = 1$.

(optional)

42. We need to prove *Grassmann's Identity*: $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$.
We will follow the hint: Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for $U \cap W$.
Then extend $\mathcal{B}$ to a basis $\mathcal{C}$ of U and a basis $\mathcal{D}$ of W.
We will show $C \cup D$ is a basis for $U + W$.

Clearly, $\operatorname{span}(C \cup D) = U + W$, so we need only show $\mathcal{C} \cup \mathcal{D}$ is linearly independent.
If $\dim U = m$ and $\dim W = n$, we have: $\mathcal{C} = \{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_m\} \cup \mathcal{B}$ and $\mathcal{D} = \{\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\} \cup \mathcal{B}$.
Since $\mathcal{C}$ and $\mathcal{D}$ are bases, $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_m\} \cap \mathcal{B} = \emptyset$ and $\{\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\} \cap \mathcal{B} = \emptyset$.
Therefore, $\mathcal{C} \cap \mathcal{D} = \mathcal{B}$. Why?
Clearly, $\mathcal{B} \subseteq \mathcal{C} \cap \mathcal{D}$, so we need only show $\mathcal{C} \cap \mathcal{D} \subseteq \mathcal{B}$.

If $\mathbf{v} \in \mathcal{C} \cap \mathcal{D}$, then $\mathbf{v} \in U \cap W$, therefore $\mathbf{v} = \sum_{i=1}^{k} c_1 \mathbf{u}_i$ because $\mathcal{B}$ is a basis for $U \cap W$.

But since $\mathbf{v} \in (\mathcal{C} \cap \mathcal{B}) \cup (\mathcal{D} \cap \mathcal{B})$, $\mathbf{v}$ must equal one of the $\mathbf{u}_i$.

So, $\mathcal{C} \cup \mathcal{D}$ is a basis for $U + W$. Therefore, $\dim(U + W) = \{$the number of vectors in $\mathcal{C} \cup \mathcal{D}\}$.

Since $\mathcal{C} \cap \mathcal{D} = \mathcal{B}$, the number of vectors in $\mathcal{C} \cup \mathcal{D}$ is
the number of vectors in $\mathcal{C} \cup \mathcal{D}$ minus the number of vectors in $\mathcal{B}$.
We subtract the number of vectors in $\mathcal{B}$ because they are counted twice in the intersection.

So, $\dim(U + W) = m + n - k = \dim U + \dim W - \dim(U \cap W)$.

Q: Why must $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_m\} \cap \mathcal{B} = \emptyset$ and $\{\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\} \cap \mathcal{B} = \emptyset$?
A: If there were any vectors in common, $\mathcal{C}$ and $\mathcal{D}$ would not be linearly independent.

Q: Why is it possible to extend $\mathcal{B}$ to a basis $\mathcal{C}$ of U and a basis $\mathcal{D}$ of W?
A: By Thm 6.10e: Any linearly independent set in V can be extended to a basis for V.

44. Clearly $\operatorname{span}(x^0 = 1, x, \ldots, x^n, \ldots) = \mathscr{P}$. Furthermore, $\sum a_i x^i = 0 \Leftrightarrow a_i = 0$ for all i.

45. $a(1 + x) + b(1 + x + x^2) + c(1) = (a + b + c) + (a + b)x + bx^2 = 0 \Rightarrow$
$a + b + c = 0$, $a + b = 0$, $b = 0 \Rightarrow$
$a = b = c = 0 \Rightarrow \mathcal{B} = \{1, 1 + x, 1 + x + x^2\}$ linearly independent $\Rightarrow$
$\mathcal{B}$ is a basis for $\mathscr{P}_2$ because $\dim \mathscr{P}_2 = 3 =$ the number of vectors in $\mathcal{B}$.

49. We will show $\text{span}(1, 1+x, 2x) = \text{span}(1, 1+x)$ and $\{1, 1+x\}$ linearly independent which will imply $\{1, 1+x\}$ is a basis for $\text{span}(1, 1+x, 2x)$.

$2x = -2(1) + 2(1+x) \Rightarrow \text{span}(1, 1+x, 2x) = \text{span}(1, 1+x)$.

$a(1) + b(1+x) = (a+b) + bx = 0 \Rightarrow a+b = 0,\ b = 0 \Rightarrow a = b = 0 \Rightarrow$
$\{1, 1+x\}$ linearly independent. Therefore, $\{1, 1+x\}$ is a basis for $\text{span}(1, 1+x, 2x)$.

27. Let $V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, V_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, V_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, V_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Then $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = a\,V_1 + b\,V_2 + c\,V_3 + d\,V_4 \Rightarrow \begin{bmatrix} a+b+c+d = 1 & b+c+d = 2 \\ c+d = 3 & d = 4 \end{bmatrix} \Rightarrow$

$A = -V_1 - V_2 - V_3 + 4\,V_4 \Rightarrow [A]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$.

29. $p(x) = 2 - x + 3x^2 = a(1) + b(1+x) + c(-1+x^2) = (a+b-c) + bx + cx^2 \Rightarrow$

$a + b - c = 2,\ b = -1,\ c = 3 \Rightarrow a = 6,\ b = -1,\ c = 3 \Rightarrow [p(x)]_{\mathcal{B}} = \begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix}$.

31. $[c_1\mathbf{u}_1 + \ldots + c_n\mathbf{u}_n]_{\mathcal{B}} = [c_1\mathbf{u}_1]_{\mathcal{B}} + \ldots + [c_n\mathbf{u}_n]_{\mathcal{B}} = c_1[\mathbf{u}_1]_{\mathcal{B}} + \ldots + c_n[\mathbf{u}_n]_{\mathcal{B}}$.

(optional)

33. Let $\text{span}(S) = \mathbb{R}^n$ and let $\mathbf{v} \in V \Rightarrow \mathbf{x} = [\mathbf{v}]_{\mathcal{B}} = \sum c_i[\mathbf{u}_i]_{\mathcal{B}} = \sum [c_i\mathbf{u}_i]_{\mathcal{B}} \Rightarrow \mathbf{v} = \sum c_i\mathbf{u}_i$ because the representation of $\mathbf{v}$ with respect to basis $\mathcal{B}$ is unique.
Therefore, $V = \text{span}(\mathbf{u}_i)$.

Let $V = \text{span}(\mathbf{u}_i)$ and let $\mathbf{x} = \begin{bmatrix} 1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n \Rightarrow \sum c_i\mathbf{u}_i = \mathbf{v} \in V$ because $V = \text{span}(\mathbf{u}_i)$.

Therefore, $\mathbf{x} = [\mathbf{v}]_{\mathcal{B}} = \sum [c_i\mathbf{u}_i]_{\mathcal{B}} = \sum c_i[\mathbf{u}_i]_{\mathcal{B}} \Rightarrow \mathbf{x} \in \text{span}(S)$.
Therefore, $\text{span}(S) = \mathbb{R}^n$.

6.3

6.3

1. (a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 = 2 \\ a_2 = 3 \end{bmatrix} \Rightarrow [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} b_1 + b_2 = 2 \\ b_1 - b_2 = 3 \end{bmatrix} \Rightarrow \begin{matrix} b_1 = \frac{5}{2} \\ b_2 = -\frac{1}{2} \end{matrix} \Rightarrow [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} \frac{5}{2} \\ -\frac{1}{2} \end{bmatrix}.$

(b) Let $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Then Theorem 6.13, which states $[\,C \,|\, B\,] \longrightarrow [\,I \,|\, P_{\mathcal{C} \leftarrow \mathcal{B}}\,]$, implies

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \Rightarrow P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

(c) $[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}} [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -\frac{1}{2} \end{bmatrix}.$

(d) Let $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Then Theorem 6.13, which states $[\,B \,|\, C\,] \longrightarrow [\,I \,|\, P_{\mathcal{B} \leftarrow \mathcal{C}}\,]$, implies

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right] \Rightarrow P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(e) $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B} \leftarrow \mathcal{C}} [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{5}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$

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DP 6.3

4. (a) $\begin{bmatrix}3\\1\\5\end{bmatrix} = a_1\begin{bmatrix}0\\1\\0\end{bmatrix} + a_2\begin{bmatrix}0\\0\\1\end{bmatrix} + a_3\begin{bmatrix}1\\0\\0\end{bmatrix} \Rightarrow \begin{matrix}a_1 = 1\\ a_2 = 5\\ a_3 = 3\end{matrix} \Rightarrow [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}1\\5\\3\end{bmatrix}.$

$\begin{bmatrix}3\\1\\5\end{bmatrix} = b_1\begin{bmatrix}1\\1\\0\end{bmatrix} + b_2\begin{bmatrix}0\\1\\1\end{bmatrix} + b_3\begin{bmatrix}1\\0\\1\end{bmatrix} \Rightarrow \begin{matrix}b_1 = -\frac{1}{2}\\ b_2 = \frac{3}{2}\\ b_3 = \frac{7}{2}\end{matrix} \Rightarrow [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix}-\frac{1}{2}\\ \frac{3}{2}\\ \frac{7}{2}\end{bmatrix}.$

(b) $\left[\begin{array}{ccc|ccc}1&0&1&0&0&1\\1&1&0&1&0&0\\0&1&1&0&1&0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\0&1&0&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\0&0&1&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{array}\right] \Rightarrow P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\ -\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{bmatrix}.$

(c) $[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\ -\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\5\\3\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\ \frac{3}{2}\\ \frac{7}{2}\end{bmatrix}.$

(d) $\left[\begin{array}{ccc|ccc}0&0&1&1&0&1\\1&0&0&1&1&0\\0&1&0&0&1&1\end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc}1&0&0&1&1&0\\0&1&0&0&1&1\\0&0&1&1&0&1\end{array}\right] \Rightarrow P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix}.$

(e) $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ \frac{3}{2}\\ \frac{7}{2}\end{bmatrix} = \begin{bmatrix}1\\5\\3\end{bmatrix}.$

5. (a) $\begin{bmatrix}2\\-1\end{bmatrix} = a_1\begin{bmatrix}1\\0\end{bmatrix} + a_2\begin{bmatrix}0\\1\end{bmatrix} \Rightarrow \begin{bmatrix}a_1 = 2\\ a_2 = -1\end{bmatrix} \Rightarrow [p(x)]_{\mathcal{B}} = \begin{bmatrix}2\\-1\end{bmatrix}.$

$\begin{bmatrix}2\\-1\end{bmatrix} = b_1\begin{bmatrix}0\\1\end{bmatrix} + b_2\begin{bmatrix}1\\1\end{bmatrix} \Rightarrow \begin{bmatrix}b_2 = 2\\ b_1 + b_2 = -1\end{bmatrix} \Rightarrow \begin{matrix}b_1 = -3\\ b_2 = 2\end{matrix} \Rightarrow [p(x)]_{\mathcal{C}} = \begin{bmatrix}-3\\2\end{bmatrix}.$

(b) $\left[\begin{array}{cc|cc}0&1&1&0\\1&1&0&1\end{array}\right] \longrightarrow \left[\begin{array}{cc|cc}1&0&-1&1\\0&1&1&0\end{array}\right] \Rightarrow P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}-1&1\\1&0\end{bmatrix}.$

(c) $[p(x)]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[p(x)]_{\mathcal{B}} = \begin{bmatrix}-1&1\\1&0\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}-3\\2\end{bmatrix}.$

(d) $\left[\begin{array}{cc|cc}1&0&0&1\\0&1&1&1\end{array}\right] = \left[\begin{array}{cc|cc}1&0&0&1\\0&1&1&1\end{array}\right] \Rightarrow P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix}0&1\\1&1\end{bmatrix}.$

(e) $[p(x)]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[p(x)]_{\mathcal{C}} = \begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}-3\\2\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}.$

7. (a) $\begin{bmatrix}1\\0\\1\end{bmatrix}=a_1\begin{bmatrix}1\\1\\1\end{bmatrix}+a_2\begin{bmatrix}0\\1\\1\end{bmatrix}+a_3\begin{bmatrix}0\\0\\1\end{bmatrix}\Rightarrow\begin{matrix}a_1=1\\a_2=-1\\a_3=1\end{matrix}\Rightarrow[p(x)]_{\mathcal{B}}=\begin{bmatrix}1\\-1\\1\end{bmatrix}.$

$\begin{bmatrix}1\\0\\1\end{bmatrix}=b_1\begin{bmatrix}1\\0\\0\end{bmatrix}+b_2\begin{bmatrix}0\\1\\0\end{bmatrix}+b_3\begin{bmatrix}0\\0\\1\end{bmatrix}\Rightarrow\begin{matrix}b_1=1\\b_2=0\\b_3=1\end{matrix}\Rightarrow[p(x)]_{\mathcal{C}}=\begin{bmatrix}1\\0\\1\end{bmatrix}.$

(b) $\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&1&1&0\\0&0&1&1&1&1\end{array}\right]\longrightarrow\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&1&1&0\\0&0&1&1&1&1\end{array}\right]\Rightarrow P_{\mathcal{C}\leftarrow\mathcal{B}}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}.$

(c) $[p(x)]_{\mathcal{C}}=P_{\mathcal{C}\leftarrow\mathcal{B}}\,[p(x)]_{\mathcal{B}}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}1\\-1\\1\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}.$

(d) $\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\1&1&0&0&1&0\\1&1&1&0&0&1\end{array}\right]\longrightarrow\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-1&1&0\\0&0&1&0&-1&1\end{array}\right]\Rightarrow P_{\mathcal{B}\leftarrow\mathcal{C}}=\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}.$

(e) $[p(x)]_{\mathcal{B}}=P_{\mathcal{B}\leftarrow\mathcal{C}}\,[p(x)]_{\mathcal{C}}=\begin{bmatrix}1&0&0\\-1&1&0\\1&-1&1\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}1\\-1\\1\end{bmatrix}.$

9. (a) $\begin{bmatrix}4\\2\\0\\-1\end{bmatrix}=a_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+a_2\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+a_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+a_4\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\Rightarrow\begin{matrix}a_1=4\\a_2=2\\a_3=0\\a_4=-1\end{matrix}\Rightarrow[A]_{\mathcal{B}}=\begin{bmatrix}4\\2\\0\\-1\end{bmatrix}.$

$\begin{bmatrix}4\\2\\0\\-1\end{bmatrix}=b_1\begin{bmatrix}1\\2\\0\\-1\end{bmatrix}+b_2\begin{bmatrix}2\\1\\1\\0\end{bmatrix}+b_3\begin{bmatrix}1\\1\\0\\1\end{bmatrix}+b_4\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\Rightarrow$

$\begin{matrix}b_1=5/2\\b_2=0\\b_3=-3\\b_4=9/2\end{matrix}\Rightarrow[A]_{\mathcal{C}}=\begin{bmatrix}5/2\\0\\-3\\9/2\end{bmatrix}.$

(b) $\left[\begin{array}{cccc|cccc}1&2&1&1&1&0&0&0\\2&1&1&0&0&1&0&0\\0&1&0&0&0&0&1&0\\-1&0&1&1&0&0&0&1\end{array}\right]\longrightarrow\left[\begin{array}{cccc|cccc}1&0&0&0&1/2&0&-1&-1/2\\0&1&0&0&0&0&1&0\\0&0&1&0&-1&1&1&1\\0&0&0&1&3/2&-1&-2&-1/2\end{array}\right]\Rightarrow$ $\cdot 1)^2)$

$P_{\mathcal{C}\leftarrow\mathcal{B}}=\begin{bmatrix}1/2&0&-1&-1/2\\0&0&1&0\\-1&1&1&1\\3/2&-1&-2&-1/2\end{bmatrix}.$

(c) $[A]_{\mathcal{C}}=P_{\mathcal{C}\leftarrow\mathcal{B}}\,[A]_{\mathcal{B}}=\begin{bmatrix}1/2&0&-1&-1/2\\0&0&1&0\\-1&1&1&1\\3/2&-1&-2&-1/2\end{bmatrix}\begin{bmatrix}4\\2\\0\\-1\end{bmatrix}=\begin{bmatrix}5/2\\0\\-3\\9/2\end{bmatrix}.$

(d) $\left[\begin{array}{cccc|cccc}1&0&0&0&1&2&1&1\\0&1&0&0&2&1&1&0\\0&0&1&0&0&1&0&0\\0&0&0&1&-1&0&1&1\end{array}\right]\longrightarrow\left[\begin{array}{cccc|cccc}1&0&0&0&1&2&1&1\\0&1&0&0&2&1&1&0\\0&0&1&0&0&1&0&0\\0&0&0&1&-1&0&1&1\end{array}\right]\Rightarrow P_{\mathcal{B}\leftarrow\mathcal{C}}=\begin{bmatrix}1&2&1&1\\2&1&1&0\\0&1&0&0\\-1&0&1&1\end{bmatrix}.$

(e) $[A]_{\mathcal{B}}=P_{\mathcal{B}\leftarrow\mathcal{C}}\,[A]_{\mathcal{C}}=\begin{bmatrix}1&2&1&1\\2&1&1&0\\0&1&0&0\\-1&0&1&1\end{bmatrix}\begin{bmatrix}5/2\\0\\-3\\9/2\end{bmatrix}=\begin{bmatrix}4\\2\\0\\-1\end{bmatrix}.$

15. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \Rightarrow [\mathbf{u}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, [\mathbf{u}_2]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$

Likewise, $\mathbf{u}_2 = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \Rightarrow \mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\} = \left\{\begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}.$

(optional)

17. $a = 1 \Rightarrow \mathcal{B} = \{1, x-1, (x-1)^2 = 1 - 2x + x^2\}$ is the Taylor Polynomial basis.

Let $\mathcal{C} = \{1, x, x^2\} \Rightarrow [p(x)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}.$

$$[B \,|\, C] \longrightarrow [I \,|\, P_{\mathcal{B}\leftarrow\mathcal{C}}] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow$$

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow [p(x)]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}\,[p(x)]_{\mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -2 \\ -8 \\ -5 \end{bmatrix} \Rightarrow$$

$p(x) = -2(1) - 8(x-1) - 5(x-1)^2.$

(optional)

21. Need only show $[\mathbf{x}]_{\mathcal{D}} = P_{\mathcal{D}\leftarrow\mathcal{C}}\, P_{\mathcal{C}\leftarrow\mathcal{B}}\, [\mathbf{x}]_{\mathcal{B}}$ since this matrix with this property is unique.

By definition, $[\mathbf{x}]_{\mathcal{D}} = P_{\mathcal{D}\leftarrow\mathcal{C}}\, [\mathbf{x}]_{\mathcal{C}}$ (1) and $[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}\, [\mathbf{x}]_{\mathcal{B}}$ (2).

Substituting (2) into (1) $\Rightarrow$ $[\mathbf{x}]_{\mathcal{D}} = P_{\mathcal{D}\leftarrow\mathcal{C}}\, P_{\mathcal{C}\leftarrow\mathcal{B}}\, [\mathbf{x}]_{\mathcal{B}}$ which is what we needed to show.