math 321

Hw09 Solutions

DP 3.6

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1. Since T is the linear transformation corresponding to matrix A, $T(\mathbf{x}) = A\mathbf{x}$. So:

$$T(\mathbf{u}) = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}, T(\mathbf{v}) = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}.$$

4. We prove T is a linear transformation by showing that $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$.

Let $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x+2y \\ 3x-4y \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Then

$$\begin{aligned} T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) &= T\left(c_1\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} -c_1y_1 - c_2y_2 \\ c_1x_1 + c_2x_2 + 2c_1y_1 + 2c_2y_2 \\ 3c_1x_1 + 3c_2x_2 - 4c_1y_1 - 4c_2y_2 \end{bmatrix} = \begin{bmatrix} -c_1y_1 \\ c_1x_1 + 2c_1y_1 \\ 3c_1x_1 - 4c_1y_1 \end{bmatrix} + \begin{bmatrix} -c_2y_2 \\ c_2x_2 + 2c_2y_2 \\ 3c_2x_2 - 4c_2y_2 \end{bmatrix} \\ &= c_1\begin{bmatrix} -y_1 \\ x_1 + 2y_1 \\ 3x_1 - 4y_1 \end{bmatrix} + c_2\begin{bmatrix} -y_2 \\ x_2 + 2y_2 \\ 3x_2 - 4y_2 \end{bmatrix} = c_1T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\ &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2). \end{aligned}$$

So, this is indeed a linear transformation.

7. We prove T is *not* a linear transformation by showing that $T(c\mathbf{v}) \neq cT(\mathbf{v})$ (property (2) fails

Let $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x^2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then

$$T(c\mathbf{v}) = T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\begin{bmatrix} cx \\ cy \end{bmatrix} = \begin{bmatrix} cy \\ c^2x^2 \end{bmatrix} \neq \begin{bmatrix} cy \\ cx^2 \end{bmatrix} = c\begin{bmatrix} y \\ cx^2 \end{bmatrix} = cT\begin{bmatrix} x \\ y \end{bmatrix} = cT(\mathbf{v})$$

Since $T(c\mathbf{v}) \neq cT(\mathbf{v})$ (property (2) fails), T is *not* a linear transformation.

Q: Is $S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y \end{bmatrix}$ linear?

A: No, by a very similar argument to the one given above.
We should suspect both T and S are *not* linear because x^2 is *not* linear.

12. As on p.212, we confirm T is a linear transformation by finding A such that $T(\mathbf{v}) = A\mathbf{v}$.

We have $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x+2y \\ 3x-4y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}x + \begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix}y = \begin{bmatrix} 0 & -1 \\ 1 & 2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

So $T = T_A$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \\ 3 & -4 \end{bmatrix}$.

15. As in Example 3.56, we confirm T is a linear transformation by finding A such that $T(\mathbf{v}) = A\vec{v}$

We have $F\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

So we identify $F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ as the matrix performing the desired transformation.

16. As in Example 3.57, we confirm T is a linear transformation by finding A such that $T(\mathbf{v}) = A\vec{v}$

We have $R\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(x-y) \\ \frac{1}{\sqrt{2}}(x+y) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} x + \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} y = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

So, $R = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is the matrix performing the desired transformation.

21. Using the formula from Example 3.58, we compute the matrix for a rotation of $-30° = 330°$.

$R_{330°}(\mathbf{e}_1) = \begin{bmatrix} \cos 330° \\ \sin 330° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}$ and $R_{330°}(\mathbf{e}_2) = \begin{bmatrix} -\sin 330° \\ \cos 330° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$,

So by Theorem 3.31, we have: $A = \begin{bmatrix} R_{330°}(\mathbf{e}_1) & R_{330°}(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.

36. A counterclockwise rotation through $60°$ is given by $T = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix}$.

A reflection in the line $y = x$ with $\mathbf{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is given by $S = \frac{1}{2}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Then, by Theorem 3.32, the composite transformation is given by

$[S \circ T] = [S]\,[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$.

$\mathcal{D}$ P 6.4

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1. $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) = T\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} = \begin{bmatrix} a+a'+b+b' & 0 \\ 0 & c+c'+d+d' \end{bmatrix}$
$= \begin{bmatrix} a+b & 0 \\ 0 & c+d \end{bmatrix} + \begin{bmatrix} a'+b' & 0 \\ 0 & c'+d' \end{bmatrix} = T\begin{bmatrix} a & b \\ c & d \end{bmatrix} + T\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$.
$T\left(\alpha\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix} = \begin{bmatrix} \alpha a + \alpha b & 0 \\ 0 & \alpha c + \alpha d \end{bmatrix} = \alpha\begin{bmatrix} a+b & 0 \\ 0 & c+d \end{bmatrix} = \alpha T\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow$
T is a linear transformation.

2. $T\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow T$ is *not* a linear transformation (it fails Theorem 6.14a).
Note this is an application of the principle $p \to q \Leftrightarrow -q \to -p$.

8. $T(0) = 1 + x + x^2 \neq \mathbf{0} \Rightarrow T(0) \neq \mathbf{0} \Rightarrow T$ is *not* a linear transformation.

9. $T((a+bx+cx^2) + (a'+b'x+c'x^2)) = T((a+a') + (b+b')x + (c+c')x^2)$
$= (a+a')+(b+b')(x+1)+(c+c')(x+1)^2 = (a+b(x+1)+c(x+1)^2)+(a'+b'(x+1)+c'(x+1)^2)$
$= T((a+bx+cx^2) + T(a'+b'x+c'x^2)$ and
$T(\alpha(a+bx+cx^2)) = T(\alpha a + \alpha bx + \alpha cx^2)) = \alpha a + \alpha b(x+1) + \alpha c(x+1)^2$
$= \alpha(a + b(x+1) + c(x+1)^2) = \alpha T(a+bx+cx^2) \Rightarrow T$ is a linear transformation.

14. Since $\begin{bmatrix} 5 \\ 2 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have:

$$T\begin{bmatrix} 5 \\ 2 \end{bmatrix} = 5T\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \\ 3 \end{bmatrix}.$$

Since $\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have:

$$T\begin{bmatrix} a \\ b \end{bmatrix} = aT\begin{bmatrix} 1 \\ 0 \end{bmatrix} + bT\begin{bmatrix} 0 \\ 1 \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + b\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} a+3b \\ 2a \\ -a+4b \end{bmatrix}.$$

21. Recall, by definition, in any vector space V, $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$. Then Theorem 6.14a $\Rightarrow$
$T(\mathbf{0}) = T(\mathbf{v} + (-\mathbf{v})) = T(\mathbf{v}) + T(-\mathbf{v}) = 0 \Rightarrow T(-\mathbf{v}) = -T(\mathbf{v})$ *by definition.*

26. $(S \circ T)(3 + 2x - x^2) = S(T(3+2x-x^2)) = S(2-2x) = 2 + (2-2)x + 2(-2)x^2 = 2 - 4x^2$.
$(S \circ T)(a + bx + cx^2) = S(T(a+bx+cx^2)) = S(b+2cx) = b + (b+2c)x + 4cx^2$.
Domain of $T = \mathscr{P}_2 =$ codomain of $S \Rightarrow$ we can compute $T \circ S$.
$(T \circ S)(a+bx) = T(S(a+bx)) = T(a + (a+b)x + 2bx^2) = (a+b) + 2(2b)x^2 = (a+b) + 4bx^2$.

DP 6.5

1. (a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\ker(T) \Leftrightarrow a = d = 0 \Rightarrow$ only (ii) $= \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$ in $\ker(T)$.

   (b) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\text{range}(T) \Leftrightarrow b = c = 0 \Rightarrow$ only (iii) $= \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$ in range(T).

   (c) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\ker(T) \Leftrightarrow a = d = 0 \Rightarrow \ker(T) = \left\{\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\right\}$.

   $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\text{range}(T) \Leftrightarrow b = c = 0 \Rightarrow \text{range}(T) = \left\{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right\}$.

3. (a) $p(x)$ in $\ker(T) \Leftrightarrow \begin{matrix} a - b = 0 \\ b + c = 0 \end{matrix} \Leftrightarrow a = b = -c \Rightarrow$ only (iii) in $\ker(T)$.

   (b) $p(x)$ in $\text{range}(T) \Leftrightarrow a,\ b,\ c \in R \Rightarrow$ (i), (ii), (iii) in range(T).

   (c) $p(x)$ in $\ker(T) \Leftrightarrow a = b = -c \Rightarrow \ker(T) = \{t + tx - tx^2\}$.

   $p(x)$ in $\text{range}(T) \Leftrightarrow a,\ b,\ c \in \mathbb{R} \Rightarrow \text{range}(T) = \mathbb{R}^2$.

4. (a) $p(x)$ in $\ker(T) \Leftrightarrow xp'(x) = x(b + 2cx) = 0 \Rightarrow b = c = 0 \Rightarrow$ only (ii) $= 2$ in $\ker(T)$.

   (b) $p(x)$ in $\text{range}(T) \Leftrightarrow p(x) = x(b + 2cx) = bx + 2cx^2 \Rightarrow a = 0, b,\ c$ free $\Rightarrow$ only (ii) $= x^2$ in range(T).

   (c) $p(x)$ in $\ker(T) \Rightarrow b = c = 0 \Rightarrow \ker(T) = \{a\}$.

   $p(x)$ in $\text{range}(T) \Rightarrow a = 0, b,\ c$ free $\Rightarrow \text{range}(T) = \{bx + cx^2\}$.

7. $\ker(T) = \{t + tx - tx^2\} \Rightarrow \{1 + x - x^2\}$ is basis $\Rightarrow \text{nullity}(T) = 1$.

   $\text{range}(T) = \mathbb{R}^2 \Rightarrow \text{rank}(T) = \dim \text{range}(T) = \dim \mathbb{R}^2 = 2$.

8. $\ker(T) = \{a\} \Rightarrow \{1\}$ is basis $\Rightarrow \text{nullity}(T) = 1$.

   $\text{range}(T) = \{bx + cx^2\} \Rightarrow \{x, x^2\}$ is basis $\Rightarrow \text{rank}(T) = 2$.

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RP 6-5

9. $A \in \ker(T) \Rightarrow = \begin{matrix} a-b=0 \\ c-d=0 \end{matrix} \Rightarrow \begin{matrix} a=b \\ c=d \end{matrix} \Rightarrow$

$\ker(T) = \left\{ \begin{bmatrix} a & a \\ c & c \end{bmatrix} \right\} \Rightarrow \ker(T) = \text{span}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right) \Rightarrow \text{nullity}(T) = \dim \ker(T) = 2.$

Therefore, $\text{rank}(T) = \dim M_{22} - \text{nullity}(T) = 4 - 2 = 2.$

10. $T(x) = \begin{bmatrix} p(0) = 0 \\ p(1) = 1 \end{bmatrix}, T(1-x) = \begin{bmatrix} p(0) = 1 - 0 = 1 \\ p(1) = 1 - 1 = 0 \end{bmatrix} \Rightarrow$

12. $A \in \ker(T) \Rightarrow AB - BA = \begin{bmatrix} c-b & d-a \\ a-d & b-c \end{bmatrix} = 0 \Rightarrow \begin{matrix} b-c=0 \\ a-d=0 \end{matrix} \Rightarrow \begin{matrix} c=b \\ d=a \end{matrix} \Rightarrow$

$\ker(T) = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \Rightarrow \text{nullity}(T) = \dim \ker(T) = 2.$

Therefore, $\text{rank}(T) = \dim M_{22} - \text{nullity}(T) = 4 - 2 = 2.$

16. (a) $\begin{bmatrix} x \\ y \end{bmatrix} \in \ker(T) \Rightarrow = \begin{matrix} x - 2y = 0 \\ 3x + y = 0 \\ x + y = 0 \end{matrix} \Rightarrow x = y = 0 \Rightarrow \ker(T) = \{\mathbf{0}\} \Rightarrow T$ is one-to-one.

(b) (a) $\Rightarrow \text{rank}(T) = \dim \mathbb{R}^2 - \text{nullity}(T) = 2 - 0 = 2 \Rightarrow \dim \text{range}(T) = 2 < \dim \mathbb{R}^3 = 3 \Rightarrow$ T is *not* onto.

Furthermore, $\dim \text{range}(T) \leq 2 < \dim \mathbb{R}^3 = 3 \Rightarrow$ no T can be onto.

18. (a) Exercise 10 $\Rightarrow \text{nullity}(T) = 1 \Rightarrow \ker(T) \neq \{\mathbf{0}\} \Rightarrow T$ is *not* one-to-one.

(b) Exercise 10 $\text{range}(T) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \mathbb{R}^2 \Rightarrow T$ is onto *by definition.*

21. $D_3 = \text{span}(E_{11}, E_{22}, E_{33}) \Rightarrow \dim D_3 = 3 = \dim \mathbb{R}^3 \Rightarrow D_3 \cong \mathbb{R}^3$.

Define $T\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Then $A \in \ker(T) \Rightarrow \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} = 0 \Rightarrow x = y = z = 0 \Rightarrow$

$\ker(T) = \{\mathbf{0}\} \Rightarrow T$ is one-to-one.

Since T is one-to-one and $\dim D_3 = \dim \mathbb{R}^3$, Theorem 6.21 implies T is onto.

22. $A \in S_3 \Rightarrow A = A^T \Rightarrow\Rightarrow \begin{matrix} a_{21} = a_{12} \\ a_{31} = a_{13} \\ a_{32} = a_{23} \end{matrix}$ and a_{11}, a_{22}, a_{33} free $\Rightarrow \dim S_3 = 6$.

Likewise, $A \in U_3 \Rightarrow \begin{matrix} a_{21} = a_{31} = a_{32} = 0 \\ a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33} \text{ free} \end{matrix} \Rightarrow \dim U_3 = 6$.

Therefore, $\dim S_3 = \dim U_3 \Rightarrow S_3 \cong U_3$.

Define $T\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$.

Then $A \in \ker(T) \Rightarrow a_{11} = a_{12} = a_{13} = a_{22} = a_{23} = a_{33} = 0 \Rightarrow \ker(T) = \{\mathbf{0}\} \Rightarrow$
T is one-to-one.

Since T is one-to-one and $\dim S_3 = \dim U_3$, Theorem 6.21 implies T is onto.

24. $\mathscr{P}_2 = \text{span}(1, x, x^2) \Rightarrow \dim \mathscr{P}_2 = 3$.

$p(x) \in W \Rightarrow p(0) = a + b(0) + c(0)^2 + d(0)^3 = 0 \Rightarrow a = 0 \Rightarrow W = \text{span}(x, x^2, x^3) \Rightarrow \dim W = 3$.

Therefore, since $\dim \mathscr{P}_2 = \dim W = 3$, $\mathscr{P}_2 \cong W$.

Define $T(a + bx + x^3) = ax + bx^2 + cx^3$. Then $p(x) \in \ker(T) \Rightarrow ax + bx^2 + cx^3 = 0 \Rightarrow$
$a = b = c = 0 \Rightarrow \ker(T) = \{\mathbf{0}\} \Rightarrow T$ is one-to-one.

Since T is one-to-one and $\dim \mathscr{P}_2 = \dim W$, Theorem 6.21 implies T is onto.

28. Need only show $p(x-2) = 0 \Rightarrow p(x) = 0$ since that implies $p(x) \in \ker(T) \Rightarrow p(x) = 0 \Rightarrow$
$\ker(T) = \{\mathbf{0}\} \Rightarrow T$ is one-to-one.

Note, this amounts proving that $\{1, x-2, (x-2)^2, \ldots, (x-2)^n\}$ is a basis for $\mathscr{P}_n$.

We will proceed by induction.

Case: $n = 1$. Then $p(x-2) = a_0 + a_1(x-2) = 0 \Rightarrow a_1 = 0 \Rightarrow a_0 = a_1 = 0 \Rightarrow p(x) = 0$.

Assume $q(x-2) = 0 \Rightarrow q(x) = 0$ for any polynomial of degree $\leq n$.
Then let $p(x) = a_0 + a_1x + \ldots + a_nx^n$ be such that $p(x-2) = 0$. We will show $p(x) = 0$.

$p(x-2) = 0 \Rightarrow a_n = 0 \Rightarrow$ the degree of $p(x) \leq n \Rightarrow p(x) = 0$ by the induction hypothesis.

Alternative proof using change-of-variable: Need only show $p(x) \in \ker(T) \Rightarrow p(x)$ for all
$x \in \mathbb{R}$ since that implies $p(x) = 0 \Rightarrow \ker(T) = \{\mathbf{0}\} \Rightarrow T$ is one-to-one.

$p(x) \in \ker(T) \Rightarrow p(x-2) = 0$ for all $x \in \mathbb{R} \Rightarrow p(y) = 0$ for all y $\in \mathbb{R}$, where $y = x - 2 \Rightarrow$
$p(x) = 0$ for all x $\in \mathbb{R}$.

33. (a) Recall L is one-to-one if and only if $L(\mathbf{v}) = \mathbf{0} \Leftrightarrow \mathbf{v} = \mathbf{0}$ because $\ker L = \{\mathbf{0}\}$.
So, we need to show $(S \circ T)(\mathbf{u}) = \mathbf{0} \Leftrightarrow \mathbf{u} = 0$.
$(S \circ T)(\mathbf{u}) = \mathbf{0} \Leftrightarrow S(T(\mathbf{u})) = \mathbf{0} \Leftrightarrow T(\mathbf{u}) = \mathbf{0} \Leftrightarrow \mathbf{u} = 0$ which shows $S \circ T$ is one-to-one.

(b) Recall L is onto for every $\mathbf{w} \in W$ there exists $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{w}$.
So, we need to show for every $\mathbf{w} \in W$ there exists $\mathbf{u} \in U$ such that $(S \circ T)(\mathbf{u}) = \mathbf{w}$.
S onto $\Rightarrow$ for every $\mathbf{w} \in W$ there exists $\mathbf{v} \in V$ such that $S(\mathbf{v}) = \mathbf{w}$.
Furthermore, T onto $\Rightarrow$ there exists $\mathbf{u} \in U$ such that $T(\mathbf{u}) = \mathbf{v} \Rightarrow$
$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u})) = S(\mathbf{v}) = \mathbf{w}$ which shows $S \circ T$ is onto.

(optional)

37. NOTE: The *null spaces* in the text should be referred to as *kernels*.
Following the hint, we will use the Rank Theorem to show $\ker(T) = \ker(T^2)$.
The Rank Theorem implies $\text{rank}(T) + \text{nullity}(T) = \text{rank}(T^2) + \text{nullity}(T^2) = \dim V$.
So since $\text{rank}(T) = \text{rank}(T^2)$, we have $\text{nullity}(T^2) = \text{nullity}(T)$.

So we will show $\ker(T) \subseteq \ker(T^2)$ and conclude that $\ker(T) = \ker(T^2)$.
Let $\mathbf{v}$ be in $\ker(T)$, then $T^2(\mathbf{v}) = T(T(\mathbf{v})) = T(\mathbf{0}) = \mathbf{0}$ which implies $\ker(T) \subseteq \ker(T^2)$.
So $\ker(T) = \ker(T^2)$ as we were to show.

Now let $\mathbf{v}$ be in $\text{range}(T) \cap \ker(T)$. We want to show $\mathbf{v} = \mathbf{0}$.
Then $\mathbf{v} = T(\mathbf{w})$ and $T(\mathbf{v}) = T(T(\mathbf{w})) = \mathbf{0}$ which implies $\mathbf{w}$ is in $\ker(T^2)$.
But $\ker(T) = \ker(T^2)$, so $\mathbf{w}$ is also in $\ker(T)$.
That is, $\mathbf{v} = T(\mathbf{w}) = \mathbf{0}$ as we were to show.

(optional)

38. Let U and V be subspaces of a finite-dimensional vector space V with $T(\mathbf{u}, \mathbf{w}) = \mathbf{u} - \mathbf{w}$.

(a) We need to show that T is a linear transformation.

$$\begin{aligned} T(\mathbf{u}_1 + \mathbf{u}_2, \mathbf{w}_1 + \mathbf{w}_2) &= (\mathbf{u}_1 + \mathbf{u}_2) - (\mathbf{w}_1 + \mathbf{w}_2) = (\mathbf{u}_1 - \mathbf{w}_1) + (\mathbf{u}_2 - \mathbf{w}_2) \\ &= T(\mathbf{u}_1, \mathbf{w}_1) + T(\mathbf{u}_2, \mathbf{w}_2). \\ T(c\mathbf{u}, c\mathbf{w}) &= c\mathbf{u} - c\mathbf{w} = c(\mathbf{u} - \mathbf{w}) = cT(\mathbf{u}, \mathbf{w}). \end{aligned}$$

(b) We need to show that $\operatorname{range}(T) = U + W$.

Recall from Exercise 48 of Section 6.1 that $U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \text{ is in } U, \mathbf{w} \text{ is in } W\}$.
Then simply note $T(\mathbf{u}, -\mathbf{w}) = \mathbf{u} - (-\mathbf{w}) = \mathbf{u} + \mathbf{w}$.
Q: Why is this sufficient?
A: Hint: Think about the fact that $-(-\mathbf{w}) = \mathbf{w}$ when substituting into T as well.

(c) We need to show that $\ker(T) \cong U \cap W$.

By Theorem 6.25, since $\ker(T)$ and $U \cap W$ are finite,
we need only show $\dim(\ker(T)) = \operatorname{nullity}(T) = \dim(U \cap W)$.
Let $\mathbf{v} = (\mathbf{u}, \mathbf{w})$ be in $\ker(T)$, then $\mathbf{u} - \mathbf{w} = \mathbf{0} \Rightarrow \mathbf{u} = \mathbf{w} = \mathbf{x}$, where $\mathbf{x}$ is in $U \cap W$.
So, we have $\mathbf{v} = (\mathbf{x}, \mathbf{x})$ where $\mathbf{x}$ is in $U \cap W$.
Therefore, if $\{\mathbf{x}_k\}$ is a basis for $U \cap W$ then $\{(\mathbf{x}_k, \mathbf{x}_k)\}$ is a basis for $\ker(T)$.
So, $\dim(\ker(T)) = \operatorname{nullity}(T) = \dim(U \cap W)$ which implies $\ker(T) \cong U \cap W$.

(d) We need to prove *Grassmann's Identity*: $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$.

From Exercise 43 in Section 6.2, we have $\dim(U \times W) = \dim U + \dim W$.
From the Rank Theorem we have: $\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(U \times W) = \dim U + \dim W$.
From parts (a) and (b), we have:
$\operatorname{rank}(T) = \dim \operatorname{range}(T) = \dim(U + W)$ and $\operatorname{nullity}(T) = \dim \ker(T) = \dim(U \cap W)$.

Substituting yields: $\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$ which implies

Grassmann's Identity: $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$ as we were to show.