math 501 HW 10 Solutions

(1)

DP 6.6

1. Directly, $T(4+2x)=2-4x$.

$[T(1)]_{\mathcal{C}}=[0-x]_{\mathcal{C}}=\begin{bmatrix}0\\-1\end{bmatrix}$, $[T(x)]_{\mathcal{C}}=[1-0x]_{\mathcal{C}}=\begin{bmatrix}1\\0\end{bmatrix} \Rightarrow [T]_{\mathcal{C}\leftarrow\mathcal{B}}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$.

So, $[T]_{\mathcal{C}\leftarrow\mathcal{B}}[4+2x]_{\mathcal{B}}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}=\begin{bmatrix}2\\-4\end{bmatrix}=[2-4x]_{\mathcal{C}}=[T(4+2x)]_{\mathcal{C}}$.

4. Directly as in Exercise 3, but since $a+bx+cx^2=(a+2b+4c)+(b+4c)(x-2)+c(x-2)^2$, we have $[a+bx+cx^2]_{\mathcal{B}}=\begin{bmatrix}a+2b+4c\\b+4c\\c\end{bmatrix}$.

$[T(1)]_{\mathcal{C}}=[1]_{\mathcal{C}}=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $[T(x+2)]_{\mathcal{C}}=[x+4]_{\mathcal{C}}=\begin{bmatrix}4\\1\\0\end{bmatrix}$,

$[T((x+2)^2)]_{\mathcal{C}}=[(x+4)^2]_{\mathcal{C}}=[16+8x+x^2]_{\mathcal{C}}=\begin{bmatrix}16\\8\\1\end{bmatrix} \Rightarrow [T]_{\mathcal{C}\leftarrow\mathcal{B}}=\begin{bmatrix}1&4&16\\0&1&8\\0&0&1\end{bmatrix}$. So:

$$\begin{aligned}[T]_{\mathcal{C}\leftarrow\mathcal{B}}[4+2x]_{\mathcal{B}}&=\begin{bmatrix}1&4&16\\0&1&8\\0&0&1\end{bmatrix}\begin{bmatrix}a+2b+4c\\b+4c\\c\end{bmatrix}=\begin{bmatrix}a+2b+4c\\b+4c\\c\end{bmatrix}_{\mathcal{C}}\\&=[(a+2b+4c)+(b+4c)x+cx^2]_{\mathcal{C}}=[a+b(x+2)+c(x+2)^2]_{\mathcal{C}}\\&=[T(a+bx+cx^2)]_{\mathcal{C}}.\end{aligned}$$

7. Directly, $T\begin{bmatrix}-7\\7\end{bmatrix}=\begin{bmatrix}7\\7\\7\end{bmatrix}$. Also, note since $\begin{bmatrix}-7\\7\end{bmatrix}=2\begin{bmatrix}1\\2\end{bmatrix}-3\begin{bmatrix}3\\-1\end{bmatrix}$, $\begin{bmatrix}-7\\7\end{bmatrix}_{\mathcal{B}}=\begin{bmatrix}2\\-3\end{bmatrix}$.

Likewise, $\left[T\begin{bmatrix}1\\2\end{bmatrix}\right]_{\mathcal{C}}=\begin{bmatrix}5\\-1\\2\end{bmatrix}_{\mathcal{C}}=\begin{bmatrix}6\\-3\\2\end{bmatrix}$, $\left[T\begin{bmatrix}3\\-1\end{bmatrix}\right]_{\mathcal{C}}=\begin{bmatrix}1\\-3\\-1\end{bmatrix}_{\mathcal{C}}=\begin{bmatrix}4\\-2\\-1\end{bmatrix} \Rightarrow$

$[T]_{\mathcal{C}\leftarrow\mathcal{B}}=\begin{bmatrix}6&4\\-3&-2\\2&-1\end{bmatrix}$.

So, $[T]_{\mathcal{C}\leftarrow\mathcal{B}}\begin{bmatrix}-7\\7\end{bmatrix}_{\mathcal{B}}=\begin{bmatrix}6&4\\-3&-2\\2&-1\end{bmatrix}\begin{bmatrix}2\\-3\end{bmatrix}=\begin{bmatrix}0\\0\\7\end{bmatrix}=\begin{bmatrix}7\\7\\7\end{bmatrix}_{\mathcal{C}}=\left[T\begin{bmatrix}-7\\7\end{bmatrix}\right]_{\mathcal{C}}$.

9. Directly $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

$[T(E_{11})]_C = [E_{11}]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, Likewise, $[T(E_{12})]_C = [E_{21}]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$,

$[T(E_{21})]_C = [E_{12}]_C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $[T(E_{22})]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow [T]_{C\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

So, $[T]_{C\leftarrow\mathcal{B}}[A]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix} = \left[\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right]_C = [T(A)]_C$.

10. Directly as in Exercise 9, but $[A]_{\mathcal{B}} = \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}$ since $\mathcal{B} = \{E_{22}, E_{21}, E_{12}, E_{11}\}$ in that order.

$[T(E_{22})]_C = [E_{22}]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ also because the *order* of the E_{ij} in C matter.

Likewise, $[T(E_{21})]_C = [E_{12}]_C == \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $[T(E_{12})]_C = [E_{21}]_C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $[T(E_{11})]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$

$[T]_{C\leftarrow\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

So, $[T]_{C\leftarrow\mathcal{B}}[A]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} = \begin{bmatrix} c \\ b \\ d \\ a \end{bmatrix} = \left[\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right]_C = [T(A)]_C$.

Again, $\begin{bmatrix} c \\ b \\ d \\ a \end{bmatrix} = \left[\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right]_C$ because of the order of E_{ij} in C.

3

(optional)

17. (a) Let $p(x) = c + dx$, then $T(p(x)) = T(c+dx) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = \begin{bmatrix} c \\ c+d \end{bmatrix} \Rightarrow$

$(S \circ T)(p(x)) = S(T(p(x)) = S(T(c+dx)) = S \begin{bmatrix} c \\ c+d \end{bmatrix} = \begin{bmatrix} c - 2(c+d) \\ 2c - (c+d) \end{bmatrix} = \begin{bmatrix} -c-2d \\ c-d \end{bmatrix} \Rightarrow$

$[(S \circ T)(1)]_{\mathcal{D} \leftarrow \mathcal{B}} = \begin{bmatrix} -1 - 2(0) \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and

$[(S \circ T)(x)]_{\mathcal{D} \leftarrow \mathcal{B}} = \begin{bmatrix} -0 - 2(1) \\ 0 - 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \Rightarrow [S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}} = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix}$.

(b) $[T(1)]_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} p(0) = 1 \\ p(1) = 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $[T(x))]_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} p(0) = 0 \\ p(1) = 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow$

$[T]_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

$[S \begin{bmatrix} 1 \\ 0 \end{bmatrix}]_{\mathcal{D} \leftarrow \mathcal{C}} = \begin{bmatrix} 1 - 2(0) \\ 2(1) - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $S \begin{bmatrix} 0 \\ 1 \end{bmatrix}]_{\mathcal{D} \leftarrow \mathcal{C}} = \begin{bmatrix} 0 - 2(1) \\ 2(0) - 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \Rightarrow$

$[S]_{\mathcal{D} \leftarrow \mathcal{C}} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$.

Therefore, $[S]_{\mathcal{D} \leftarrow \mathcal{B}} = [S]_{\mathcal{D} \leftarrow \mathcal{C}} [T]_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix}$ as in (a).

(optional

19. In Exercise 1, since both bases were already standard, we have:

$[T]_{\mathcal{E}' \leftarrow \mathcal{E}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ invertible $\Rightarrow T$ invertible and

$[T^{-1}(a+bx)]_{\mathcal{E}' \leftarrow \mathcal{E}} = ([T]_{\mathcal{E}' \leftarrow \mathcal{E}})^{-1} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} \Rightarrow$

$T^{-1}(a+bx) = -b + ax$.

(optional)

40. By Exercise 39, $A = [T]_{\mathcal{C} \leftarrow \mathcal{B}} \Leftrightarrow A[\mathbf{b}_i]_{\mathcal{B}} = A\mathbf{e}_i = \mathbf{a}_i = [T(\mathbf{b}_i)]_{\mathcal{C}}$.
Also, $\text{nullity}(A) = \dim(\text{null}(A) = \{\mathbf{v} : A\mathbf{v} = \mathbf{0}\})$ and $\text{nullity}(T) = \dim \ker(T) = \{T(\mathbf{v}) : T(\mathbf{v}) = \mathbf{0}\})$.
Will show $\{\mathbf{b}_1, \ldots, \mathbf{b}_m\} \subseteq \mathcal{B}$ is a basis for $\text{null}(A) \Leftrightarrow \{T(\mathbf{b}_1), \ldots, T(\mathbf{b}_m)\}$ is a basis for $\ker(T)$.
Furthermore, $\mathbf{a}_i = A[\mathbf{b}_i]_{\mathcal{B}} = [T(\mathbf{b}_i)]_{\mathcal{C}}$ linearly independent $\Leftrightarrow T(\mathbf{b}_i)$ linearly independent.
So we need only show $\text{null}(A) = \text{span}(\mathbf{b}_1, \ldots, \mathbf{b}_m) \Leftrightarrow \ker(T) = \text{span}(T(\mathbf{b}_1), \ldots, T(\mathbf{b}_m))$.
Section 6.2 Theorem 6.7 $\Rightarrow \mathbf{0} = A[\mathbf{b}_i]_{\mathcal{B}} = [T(\mathbf{b}_i)]_{\mathcal{C}} \Leftrightarrow T(\mathbf{b}_i) = \mathbf{0} \Rightarrow$
$\text{null}(A) = \text{span}(\mathbf{b}_1, \ldots, \mathbf{b}_m) \Leftrightarrow \ker(T) = \text{span}(T(\mathbf{b}_1), \ldots, T(\mathbf{b}_m)) \Rightarrow \text{nullity}(T) = \text{nullity}(A)$.

(optional)

44. We will show $[\mathbf{x}]_{\mathcal{C}'} = P_{\mathcal{C}' \leftarrow \mathcal{C}}[T]_{\mathcal{C} \leftarrow \mathcal{B}} P_{\mathcal{B} \leftarrow \mathcal{B}'}[\mathbf{x}]_{\mathcal{B}'}$ which will imply
$[T]_{\mathcal{C}' \leftarrow \mathcal{B}'} = P_{\mathcal{C}' \leftarrow \mathcal{C}}[T]_{\mathcal{C} \leftarrow \mathcal{B}} P_{\mathcal{B} \leftarrow \mathcal{B}'}$ since this matrix with this property is unique.
$P_{\mathcal{C}' \leftarrow \mathcal{C}}[T]_{\mathcal{C} \leftarrow \mathcal{B}}(P_{\mathcal{B} \leftarrow \mathcal{B}'}[\mathbf{x}]_{\mathcal{B}'}) = P_{\mathcal{C}' \leftarrow \mathcal{C}}([T]_{\mathcal{C} \leftarrow \mathcal{B}}[\mathbf{x}]_{\mathcal{B}}) = P_{\mathcal{C}' \leftarrow \mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = [\mathbf{x}]_{\mathcal{C}'} \Rightarrow$
$[T]_{\mathcal{C}' \leftarrow \mathcal{B}'} = P_{\mathcal{C}' \leftarrow \mathcal{C}}[T]_{\mathcal{C} \leftarrow \mathcal{B}} P_{\mathcal{B} \leftarrow \mathcal{B}'}$ as was to be shown.

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4.2

2. As in Example 4.8, we compute $\det A$ by expanding along the first *row* and the first *column*

$$\textit{row}: \quad \begin{vmatrix} 0 & 1 & -1 \\ 2 & 3 & -2 \\ -1 & 3 & 0 \end{vmatrix} = -1\begin{vmatrix} 2 & -2 \\ -1 & 0 \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ -1 & 3 \end{vmatrix} = -1(-2) - 1(9) = -7$$

$$\textit{column}: \quad \begin{vmatrix} 0 & 1 & -1 \\ 2 & 3 & -2 \\ -1 & 3 & 0 \end{vmatrix} = -2\begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -2(3) - 1(1) = -7$$

7. As in Example 4.10, we choose a row or column that minimizes the number of calculations. Since $\mathbf{A}_3 = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$ contains two zeroes, $\det A = a_{31}C_{31} = a_{31}(-1)^{3+1}\det A_{31} = 3\det A_{31}$.

$$\textit{row 3}: \quad \begin{vmatrix} 5 & 2 & 2 \\ -1 & 1 & 2 \\ 3 & 0 & 0 \end{vmatrix} = 3\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 3(2) = 6$$

Q: What should we look for when choosing a row or column to expand along?
A: A row or column with the maximum number of zeroes. Why?
The maximum number of zeroes minimizes the number of cofactors we have to compute.

8. As in Example 4.10, we choose a row or column that minimizes the number of calculations. Since $\mathbf{A}_2 = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$ contains one zero, $\det A = -2\det A_{21} - 1\det A_{23}$.

$$\textit{row 2}: \quad \begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 3 & -2 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -2(-1) - 1(-5) = 7$$

Q: Why is the coefficient of $\det A_{21}$ equal to -2 instead of 2?
A: Because the cofactor $C_{21} = (-1)^{2+1}\det A_{21} = -\det A_{21}$.
Q: Why is the coefficient of $\det A_{23}$ equal to -1 instead of 1?

As in Example 4.10, we choose a row or column that minimizes the number of calculations

Since $\mathbf{a}_1 = \begin{bmatrix} \cos\theta \\ 0 \\ 0 \end{bmatrix}$ contains two zeroes, $\det A = \cos\theta \det A_{11}$.

$$\textit{col 1}: \quad \begin{vmatrix} \cos\theta & \sin\theta & \tan\theta \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{vmatrix} = \cos\theta\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos\theta(\cos^2\theta + \sin^2\theta) = \cos\theta$$

17. Following the method of Example 4.9, we have:

$$\begin{array}{ccccc} & & 0 & -2 & 2 \\ 1 & 1 & -1 & 1 & 1 \\ 2 & 0 & 1 & 2 & 0 \\ 3 & -2 & 1 & 3 & -2 \\ & & 0 & 3 & 4 \end{array}$$

Adding the three products at the bottom and subtracting the three products at the top gives $\det A = 0 + 3 + 4 - 0 - (-2) - 2 = 7$.

21. We use induction to prove Theorem 4.2: if A is triangular then $\det A = a_{11}a_{22}\cdots a_{nn}$.
See Appendix B for discussion and examples of *Mathematical Induction.*
Since Theorem 4.10 asserts $\det A = \det A^T$, we can assume A is upper triangular.

1: If A is 1×1, then $A = [a_{11}]$ so $\det A = \det([a_{11}]) = a_{11}$.
This is obvious, so there is nothing to show.

n: If A is $n \times n$ and upper triangular, then $\det A = a_{11}a_{22}\cdots a_{nn}$.
This is the induction hypothesis.

$n+1$: If A is $(n+1) \times (n+1)$ and upper triangular, then $\det A = a_{11}a_{22}\cdots a_{nn}a_{n+1n+1}$.
This is the statement we must prove using the induction hypothesis.

$$\det A \overset{\substack{A\text{ is upper}\\ \text{triangular}}}{=} a_{n+1n+1}\det A_{n+1n+1} \overset{\substack{\text{by}\\ \text{induction}}}{=} a_{n+1n+1}(a_{11}a_{22}\cdots a_{nn}) = a_{11}a_{22}\cdots a_{n+1n+1}$$

We have proven (by induction) that if A is $n \times n$ and triangular, then $\det A = a_{11}a_{22}\cdots a_{nn}$

Q: Why does the fact that $\det A = \det A^T$ allow us to assume A is upper triangular?
A: Because if A is lower triangular, then A^T is upper triangular.
So, $\det A = \det A^T = a_{11}a_{22}\cdots a_{nn}$ because $[A]_{ii} = [A^T]_{ii}$.
That is, the diagonal entries of A and A^T are equal.

Q: Why does A being upper triangular imply $\det A = a_{n+1n+1}\det A_{n+1n+1}$?
A: Because if A is upper triangular, then row $n+1 = \mathbf{A}_{n+1} = \begin{bmatrix} 0 & 0 & \cdots & a_{n+1n+1} \end{bmatrix}$.
So when we expand along this row we have: $\det A = a_{n+1n+1}\det A_{n+1n+1}$.

Q: Why is the coefficient of $\det A_{n+1n+1}$ equal to a_{n+1n+1} instead of $-a_{n+1n+1}$?
A: The cofactor $C_{n+1n+1} = (-1)^{2n+2}\det A_{n+1n+1} = \det A_{n+1n+1}$ because $2n+2$ is even.

Q: Why do we get to apply the induction hypothesis to the matrix A_{n+1n+1}?
A: Since A_{n+1n+1} is created by removing the $n+1$ row and the $n+1$ column of A,
A_{n+1n+1} is $n \times n$ and upper triangular. Why is this obvious?
Since A is $(n+1) \times (n+1)$, A_{n+1n+1} (created by removing a row and column) is $n \times n$
Since A is upper triangular, $[A]_{ij} = 0 = [A_{n+1n+1}]_{ij}$ for $i > j$ when $i, j \le n$.
That is, A_{n+1n+1} is upper triangular.

All these details support the proof given above. We should carefully investigate them all.
In general, we should be critical of our reasoning and actively seek out oversights.

22. As in Example 4.13, we use Theorem 4.3 to track adjustments to $\det A$ required $A \longrightarrow U$.

$$\det A = \begin{vmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} \overset{R_2-5R_1}{=} \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & -14 \\ 0 & 1 & 2 \end{vmatrix} \overset{R_3-R_2}{=} \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & -14 \\ 0 & 0 & 16 \end{vmatrix} = 1 \cdot 1 \cdot 16 = 16 = \det U$$

Q: Even though row operations were used in $A \longrightarrow U$, we still have $\det A = \det U$. Why?
A: Both row operations were of the form $R_i + kRj$. So what does Theorem 4.3 say?
Part f. asserts a row operation like $R_i + kRj$ does not change the value of the determinant

26. Since $\mathbf{A}_3 = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = 2\mathbf{A}_1$, we have $\det A = 0$.

Q: Does Theorem 4.3 imply if $\mathbf{A}_i = k\mathbf{A}_j$ then $\det A = 0$?

A: Yes. How? Part f. asserts if $A \xrightarrow{R_i - kR_j} B$, then $\det A = \det B$.
But $\mathbf{B}_i$ (row i of matrix B) is zero, so part c. says $\det B = \det A = 0$.

Q: Does the same hold true for columns? That is, does $\mathbf{a}_i = k\mathbf{a}_j$ imply $\det A = 0$?
A: Yes. Since $\det A = \det A^T$ this conclusion holds for both rows and columns.

Q: How might we word this conclusion as part g. (so to speak) of Theorem 4.3?

Q: These two statements are specific cases of what Theorem from this section?
A: Theorem 4.6 which asserts A is invertible if and only if $\det A \neq 0$.

Q: So what does Theorem 4.6 imply if A is *not* invertible?
A: Theorem 4.6 implies A is not invertible if and only if $\det A = 0$.

Q: How is this a generalization of the statements we have just proven?
A: We have shown if one row of A is a multiple of another row, then $\det A = 0$.
But we know that if one row of A is a multiple of another row, then A is not invertible.
Theorem 4.6 implies if there is *any* dependence relation among the rows then $\det A = 0$.
Why? Because the existence of any such dependence relation implies A is not invertible

Q: Does Theorem 4.6 imply $\det A = 0$ if there is a dependence relation among the columns?
A: Yes. Since $\det A = \det A^T$ this conclusion holds for both rows and columns.

27. Since A is triangular, we have $\det A = a_{11}a_{22}a_{33} = (3)(-2)(4) = -24$.

Q: Which Theorem from this Section did we have employ in reaching this conclusion?
A: Theorem 4.2.

Q: How does the proof of this Theorem in Exercise 21 suggest solving this problem directly?
A: By expanding along row 3. So:

$$\text{row 3: } \begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 4 \end{vmatrix} = 4\begin{vmatrix} 3 & 1 \\ 0 & -2 \end{vmatrix} = 4(3)(-2) = -24$$

Note: We should apply our proofs to specific examples to see if they make sense and work.
Also: A_{33} is 2×2 and upper triangular as it should be according to proof in Exercise 21.

Q: Since $A \longrightarrow I$ (obviously), why is $\det A = -24 \neq 1 = \det I$?
A: For $A \longrightarrow I$ we have to multiply row 1 by $\frac{1}{3}$, row 2 by $-\frac{1}{2}$, and row 3 by $\frac{1}{4}$. What does that tell us? We have to do the same thing to $\det A$.
Therefore $\det I = \left(\frac{1}{3}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{4}\right) \det A = -\frac{1}{24}(-24) = 1$.

35. Let A be the matrix given at the beginning of Exercises 35 – 40 with $\det A = 4$.
Let B be the matrix given in this Exercise which is derived from A. So:
Since $A \xrightarrow{2R_1} B$, $\det B = 2\det A = 2(4) = 8$.

Q: What Theorem from this section supports our conclusion above that $\det B = 2\det A$?

A: Theorem 4.3 part d. which asserts if $A \xrightarrow{kR_i} B$, then $\det B = k\det A$.

36. Let A be the matrix given at the beginning of Exercises 35 – 40 with $\det A = 4$.
Let D be the matrix given in this Exercise which is derived from A in 3 steps.
$A \xrightarrow{3C_1} B \xrightarrow{-C_2} C \xrightarrow{2C_3} D$, so $\det D = 2\det C = 2(-\det B) = 2(-(3\det A)) = -6\det A = -24$.

Q: How might we generalize this result?

A: If $A \xrightarrow{bC_i} B \xrightarrow{cC_j} C \xrightarrow{dC_k} D$, $\det D = bcd(\det A)$.

Q: Does this result hold for rows? If $A \xrightarrow{bR_i} B \xrightarrow{cR_j} C \xrightarrow{dR_k} D$, does $\det D = bcd(\det A)$?
A: Yes. Since $\det A = \det A^T$ this result holds for both rows and columns.

40. Let A be the matrix given at the beginning of Exercises 35 – 40 with $\det A = 4$.
Let C be the matrix given in this Exercise which is derived from A in 2 steps.
Since $A \xrightarrow{2R_2} B \xrightarrow{R_2-3R_3} C$, we have $\det C = \det B = 2(\det A) = 2(4) = 8$.

Q: How might we generalize this result?

A: If $A \xrightarrow{bR_i} B \xrightarrow{R_j-cR_k} C$, $\det C = b\det A$. This is precisely Theorem 4.3, parts d. and f.

Q: Does this result hold for columns? If $A \xrightarrow{bC_i} B \xrightarrow{C_j-cC_k} C$, does $\det C = b\det A$?
A: Yes. Since $\det A = \det A^T$ this result holds for both rows and columns.

41. We will first prove Theorem 4.3 part a. for rows and then for columns.

row: If A has a zero row, then $\det A = 0$. That is, if $\mathbf{A}_i = \mathbf{0}$, then $\det A = 0$.

By Theorem 4.1, $\det A = \sum_{j=1}^{n} a_{ij}C_{ij}$, where $\mathbf{A}_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$.

If $\mathbf{A}_i = \mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$, then $a_{ij} = 0$ for all j. So, $\det A = \sum_{j=1}^{n} 0(C_{ij}) = 0$.

col: If A has a zero column, then $\det A = 0$. That is, if $\mathbf{a}_i = \mathbf{0}$, then $\det A = 0$.

By Theorem 4.1, $\det A = \sum_{i=1}^{n} a_{ij}C_{ij}$, where $\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$.

If $\mathbf{a}_j = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, then $a_{ij} = 0$ for all i. So, $\det A = \sum_{i=1}^{n} 0(C_{ij}) = 0$.

Note: Our proof is trivial because the Laplace Expansion does all the work.

Q: How might we state our proof and conclusions about rows in words?
A: When we expand along a zero row, the coefficient of every cofactor is zero so $\det A = 0$
This illustrates an important point: symbols are essential to an elegant proof,
but stating conclusions (loosely) in language is essential to solid understanding.

Q: Can we create a similar statement for our proof and conclusions about columns?

42. We will first prove Theorem 4.3(f) for rows and then for columns.

row: If $A \xrightarrow{R_i+kR_j} C$, then $\det C = \det A$.

We will prove this using part e. of Theorem 4.3. That is, we will define B such that A, B, and C are identical except that $\mathbf{C}_i = \mathbf{A}_i + \mathbf{B}_i$.
Under these conditions on A, B, and C part e. asserts $\det C = \det A + \det B$.
We will then show $\det B = 0$, so we can conclude $\det C = \det A$.

Since $A \xrightarrow{R_i+kR_j} C$ implies $C = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_i + k\mathbf{A}_j \\ \vdots \\ \mathbf{A}_n \end{bmatrix}$, we let $B = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ k\mathbf{A}_j \\ \vdots \\ \mathbf{A}_n \end{bmatrix}$.

Then $\mathbf{A}_r = \mathbf{B}_r = \mathbf{C}_r$ when $r \neq i$ and row i of C is $\mathbf{C}_i = \mathbf{A}_i + k\mathbf{A}_j = \mathbf{A}_i + \mathbf{B}_i$ as required.
So part e. of Theorem 4.3. asserts $\det C = \det A + \det B$.

Next we will show that $\det B = 0$. (Note: row i of B, $\mathbf{B}_i = k\mathbf{A}_j$).
We will prove this using parts c. and d. of Theorem 4.3. That is, we will define D such that D has two identical rows so part c. will imply $\det D = 0$ and such that $D \xrightarrow{kR_i} B$ so part d. will imply $\det B = k \det D = k(0) = 0$.

Since $B = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ k\mathbf{A}_j \\ \vdots \\ \mathbf{A}_n \end{bmatrix}$, we let $D = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_j \\ \vdots \\ \mathbf{A}_n \end{bmatrix}$ so $\mathbf{D}_i = \mathbf{D}_j = \mathbf{A}_j$ and $\mathbf{B}_i = k\mathbf{D}_i$ as required.

So parts c. and d. assert $\det B \overset{\mathbf{B}_i = k\mathbf{D}_i}{=} k \det D \overset{\mathbf{D}_i = \mathbf{D}_j}{=} k(0) = 0$.

Therefore, we conclude $\det C \overset{\mathbf{C}_i = \mathbf{A}_i + \mathbf{B}_i}{=} \det A + \det B \overset{\det B = 0}{=} \det A$ as required.

col: If $A \xrightarrow{C_i+kC_j} C$, then $\det C = \det A$.

Since $A \xrightarrow{C_i+kC_j} C$ implies $C = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_i + k\mathbf{a}_j & \cdots & \mathbf{a}_n \end{bmatrix}$,
we let $B = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & k\mathbf{a}_j & \cdots & \mathbf{a}_n \end{bmatrix}$.
Then $\mathbf{a}_r = \mathbf{b}_r = \mathbf{c}_r$ when $r \neq i$ and column i of C is $\mathbf{c}_i = \mathbf{a}_i + k\mathbf{a}_j = \mathbf{a}_i + \mathbf{b}_i$ as required.
So part e. of Theorem 4.3. asserts $\det C = \det A + \det B$.

Next we will show that $\det B = 0$. (Note: column i of B, $\mathbf{a}_i = k\mathbf{a}_j$).
Since $B = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & k\mathbf{a}_j & \cdots & \mathbf{a}_n \end{bmatrix}$, we let $D = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_j & \cdots & \mathbf{a}_n \end{bmatrix}$.
So $\mathbf{d}_i = \mathbf{d}_j = \mathbf{a}_j$ and $\mathbf{b}_i = k\mathbf{d}_i$ as required.

So parts c. and d. assert $\det B \overset{\mathbf{b}_i = k\mathbf{d}_i}{=} k \det D \overset{\mathbf{d}_i = \mathbf{d}_j}{=} k(0) = 0$.

Therefore, we conclude $\det C \overset{\mathbf{c}_i = \mathbf{a}_i + \mathbf{b}_i}{=} \det A + \det B \overset{\det B = 0}{=} \det A$ as required.

44. To prove Theorem 4.7, we need to show if $B = kA$, then $\det B = k^n \det A$.

We will begin by proving the following slightly more general result by induction:
If the first m rows of A have been multiplied by k to create B then $\det B = k^m \det A$.

1: If the first row of A has been multiplied by k to create B, then $\det B = k \det A$.
But if $A \xrightarrow{kR_1} B$, Theorem 4.3(d) implies $\det B = k \det A$.

$$\text{That is, if } B = \begin{bmatrix} k\mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_n \end{bmatrix}, \text{ then } \det(B) \overset{\substack{R_1 \to kR_1 \\ \text{Thm 4.3(d)}}}{=} k \det A$$

r: If the first r rows of A have been multiplied by k to create B, then $\det B = k^r \det A$.

$$\text{That is, if } B = \begin{bmatrix} k\mathbf{A}_1 \\ k\mathbf{A}_2 \\ \vdots \\ k\mathbf{A}_r \\ \mathbf{A}_{r+1} \\ \vdots \\ \mathbf{A}_n \end{bmatrix}, \text{ then } \det B = k^r \det A$$

This is the induction hypothesis so there is nothing to show.

$r+1$: If the first $r+1$ rows of A have been multiplied by k to create B, then $\det B = k^{r+1} \det A$.
This is the key step we must show using the induction hypothesis.

$$\text{That is, if } B = \begin{bmatrix} k\mathbf{A}_1 \\ k\mathbf{A}_2 \\ \vdots \\ k\mathbf{A}_r \\ k\mathbf{A}_{r+1} \\ \mathbf{A}_{r+2} \\ \vdots \\ \mathbf{A}_n \end{bmatrix}, \text{ we must show } \det B = k^{r+1} \det A. \text{ So let } C = \begin{bmatrix} k\mathbf{A}_1 \\ k\mathbf{A}_2 \\ \vdots \\ k\mathbf{A}_r \\ \mathbf{A}_{r+1} \\ \mathbf{A}_{r+2} \\ \vdots \\ \mathbf{A}_n \end{bmatrix}.$$

$$\text{Then } \det B \overset{\substack{R_{r+1} \to kR_{r+1} \\ \text{Thm 4.3(f)}}}{=} k \det C \overset{\substack{\text{by} \\ \text{induction}}}{=} k(k^r \det A) = k^{r+1} \det A$$

So if the first m rows of A have been multiplied by k to create B, then $\det B = k^m \det A$.

$$\text{Therefore, if } B = kA = \begin{bmatrix} k\mathbf{A}_1 \\ k\mathbf{A}_2 \\ \vdots \\ k\mathbf{A}_n \end{bmatrix}, \text{ then } \det A = k^n \det A \text{ as we were to show.}$$

47. We use Theorem 4.8, $\det(AB) = (\det A)(\det B)$, and the given values to compute $\det(AB)$

$$\det(AB) \overset{\text{Thm 4.8}}{=} (\det A)(\det B) \overset{\text{givens}}{=} (3)(-2) = -6$$

Q: How might we state the conclusion $\det(AB) = (\det A)(\det B)$ in words?
A: The determinant of the product equals the product of the determinant.

48. We use Theorem 4.8 and $\det A = 3$ to compute $\det(B^{-1}A)$.

$$\det(A^2) = \det(A \cdot A) \overset{\text{Thm 4.8}}{=} (\det A)(\det A) = (\det A)^2 \overset{\text{givens}}{=} (3)^2 = 9$$

Q: How might we state the conclusion $\det(A^2) = (\det A)^2$ in words?
A: The determinant of the square equals the square of the determinant.

49. We use Theorems 4.8, 4.9 and $\det A = 3$, $\det B = -2$ to compute $\det(B^{-1}A)$.

$$\det(B^{-1}A) \overset{\text{Thm 4.8}}{=} (\det(B^{-1}))(\det A) \overset{\text{Thm 4.9}}{=} \left(\tfrac{1}{\det B}\right)(\det A) \overset{\text{givens}}{=} (-\tfrac{1}{2})(3) = -\tfrac{3}{2}$$

51. We use Theorem 4.7 with $k = 3$, Theorem 4.10, and the givens to compute $\det(3B^T)$.

$$\det(3B^T) \overset{\substack{k=3\\ \text{Thm 4.7}}}{=} 3^n \det(B^T) \overset{\text{Thm 4.10}}{=} 3^n \det B \overset{\text{givens}}{=} 3^n(-2) = -2 \cdot 3^n$$

54. We use Exercise 53, associativity, and $MM^{-1} = I$ to prove $\det(B^{-1}AB) = \det A$.

$$\det(B^{-1}AB) \overset{\text{assoc}}{=} \det(B^{-1}(AB)) \overset{\text{Ex 53}}{=} \det((AB)B^{-1}) \overset{\text{assoc}}{=} \det(A(BB^{-1})) \overset{MM^{-1}=I}{=} \det A$$

Q: What is the key insight to take away from this exercise?
A: That it is important to justify each step and pay attention to the details.

Q: Could we have proven this equality directly as well? How?
A: Use Thms 4.8, 4.10, and $\left(\frac{1}{k}\right) k = 1$ to prove $\det(B^{-1}AB) = \det A$.

$$\det(B^{-1}AB) \overset{\text{Thm 4.8}}{=} (\det(B^{-1}))(\det A)(\det B) \overset{\text{Thm 4.10}}{=} \left(\tfrac{1}{\det B}\right)(\det A)(\det B) \overset{(1/k)k=1}{=} \det A$$

Q: What detail do we need to prove to complete the proof just given?
A: We need to show $\det(ABC) = (\det A)(\det B)(\det C)$.

Q: What are the strengths and weaknesses of the two proofs given?

Q: Could we construct a proof based on the elementary matrices?
A: See Section 3.3 where B and B^{-1} are constructed from elementary matrices.

(optional)

56. First we use induction on Theorem 4.8 to show that $\det(A^m) = (\det A)^m$.
Then we use that fact to find all possible values of $\det A$ when $A^m = O$.

1: $\det(A^1) = (\det A)^1$
This is obvious, so there is nothing to show.

k: $\det(A^k) = (\det A)^k$
This is the induction hypothesis, so there is nothing to show.

$k+1$: $\det(A^{k+1}) = (\det A)^{k+1}$
This is the statement we must prove using the induction on Theorem 4.8.

$$\det(A^{k+1}) = \det(AA^k) \overset{\text{Thm 4.8}}{=} \det A \det(A^k) \overset{\text{induction}}{=} \det A(\det A)^k = (\det A)^{k+1}$$

Therefore, $\det(A^m) = (\det A)^m$ as we were to show.

Since $(\det A)^m = \det(A^m) \overset{\text{given}}{=} \det O \overset{\text{Thm 4.3(a)}}{=} 0$, the only possible value for $\det A$ is zero.

57. We solve the system using Theorem 4.11 (Cramer's Rule). So we have:

$\det A = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$, $\det(A_1(\mathbf{b})) = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$, $\det(A_2(\mathbf{b})) = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$.

By Cramer's Rule, $x = \dfrac{\det(A_1(\mathbf{b}))}{\det A} = \dfrac{-3}{-2} = \dfrac{3}{2}$ and $y = \dfrac{\det(A_2(\mathbf{b}))}{\det A} = \dfrac{1}{-2} = -\dfrac{1}{2}$.